HW 2 due 2/25

Effective interest rate $= \frac{i^{(m)}}{m} = r$

Cohort Life Table

Age x	l_x	d_x
0	100000	2625
1	97371	141
2	97230	107
	↓	
78	31460	2803

Note: $l_0 \equiv$ radix

$d_x \equiv l_x - l_{x+1}$

$S_0(t) =$ fraction of the initial cohort surviving to $t>0$ $(t \in \mathbb{R})$

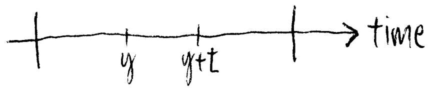
Survival function $S(t)$

$S(x) = \frac{l_x}{l_0}$ exact for $x \in \mathbb{Z}_+$ ~~$\cup \{0\}$~~
positive integers

$\forall y, t > 0$

$$\boxed{S(y) - S(y+t)} \to \frac{l_{y+t}}{l_0}$$

$\frac{l_y}{l_0}$



$(y, y+t]$

For $x, k \in \mathbb{Z}_+$

$$\boxed{\frac{S(x) - S(x+k)}{S(x)}} = \frac{l_x - l_{x+k}}{l_x}$$

$S(t) \in [0,1]$ $\quad \begin{cases} S(0) = 1 \\ S(\omega) = 0 \end{cases}$

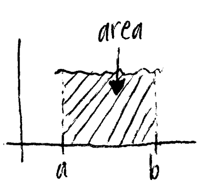
$S(t) \sim P_i\{\text{single individual surviving to } t\}$

$\boxed{f(t) = -S'(t)}$ \\ $F(t) = 1 - S(t)$

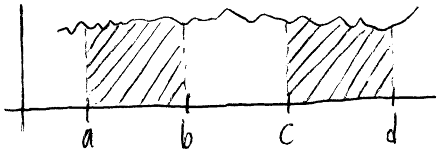
Fund. Thm.

$Pr\{\text{life age 0 dies between } 0<a<b\}$

$= S(a) - S(b) = \int_a^b [-S'(t)]\,dt = \int_a^b f(t)\,dt$



$A \subseteq \mathbb{R}_+ \to \int_A f(t)\,dt = Pr(A)$



$(a,b] = A \qquad (c,d] = B$

$A \cap B = \emptyset$

$Pr(A \cup B) = Pr(A) + Pr(B)$

Random Variable (RV)

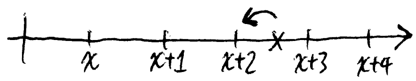
$X: \Omega \Rightarrow \mathbb{R}$ such that $\{s \in \Omega : X(s) \in (a,b)\}$ is an event

Prob. Distribution of X

$\forall (a,b]$: value of X $\quad Pr(a < X < b) \equiv Pr(s \in \Omega : X(s) \in (a,b])$

T = age of the element of the cohort that dies

Next, suppose $Y = g(T)$ specifies a contractual payment to be paid at time of death & which depends only on the integer part of T, $[T]$ (e.g. $[57.32] = 57$)



What is the average value of Y over the cohort?

Ans: $d_x = l_x - l_{x+1}$ is the fraction (out of l_0)

$\Rightarrow$ total (all cohort) $= \sum_x (l_x - l_{x+1})\, g(x)$

$\Rightarrow$ Avg. payment $(Y = g(T) = g([T]))$

$\Rightarrow \sum_x (l_x - l_{x+1})\, g(x) / l_0$

$= \sum_x (S(x) - S(x+1))\, g(x)$

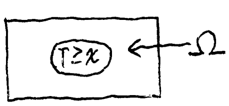
$= \sum_x \int_x^{x+1} f(t) g(t)\, dt = \int_0^\infty f(t) g(t)\, dt = E[Y]$

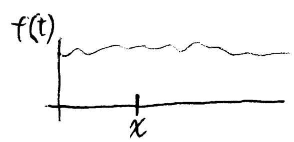
= total contingent payment over the cohort...

$\equiv$ defn. expectation of $Y = g(T)$

Same idea Restricted Population

$\rightarrow$ lives aged x

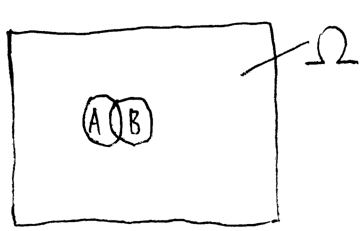




$$\int_0^w f(t)\, dt = 1 \qquad \int_x^w f(t)\, dt < 1 \qquad \frac{1}{S(x)} \int_x^w f(t)\, dt = 1$$

$$\boxed{f(t) \longleftarrow \frac{f(t)}{S(x)}}$$

replace $f(t)$ by $\frac{f(t)}{S(x)}$

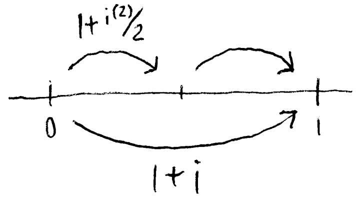


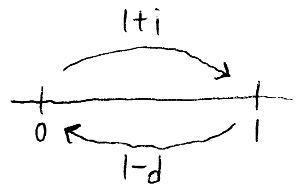
$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Review Ch 1

$i \equiv i_{eff}$

$1+i = \left(1+\frac{i^{(m)}}{m}\right)^m \;\; <-> \;\; i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$





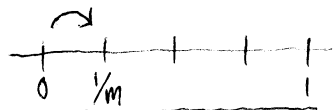
$(1+i)(1-d) = 1 \;\rightarrow\; 1+i-d-id = 1 \;\rightarrow\; i-d = id$

$1-d = \frac{1}{1+i} \equiv V \qquad\qquad \rightarrow d = i - id$

$= i(1-d)$

$= iV$

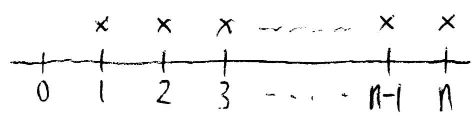
$\boxed{d = iV}$



$\boxed{1+ i^{(m)}/m = 1 - d^{(m)}/m}$

Contracts

(m=1)



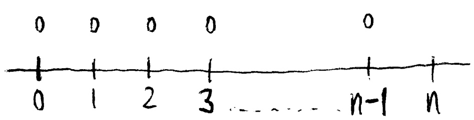
\$1

Immediate annuity $= a_{\overline{n}|} = V + V^2 + \dots + V^n = V(1+V+\dots+V^{n-1}) = V\left(\frac{1-V^n}{1-V}\right) = V\left(\frac{1-V^n}{\frac{1+i}{1+i} - \frac{1}{1+i}}\right) = V\frac{1-V^n}{i/(1+i)} = \frac{V}{\frac{1}{1+i}} \cdot \frac{1-V^n}{i} = \boxed{\frac{1-V^n}{i}}$

$a_{\overline{n}|} = (1+V+\dots+V^n) - 1 = \frac{1-V^{n+1}}{1-V} - \frac{1-V}{1-V} = \frac{V-V^{n+1}}{1-V} = V\left(\frac{1-V^n}{1-V}\right)$

Immediate perpetuity $(n=\infty)$ $= a_{\overline{\infty}|} = \frac{1}{i}$

(m=1)



\$1 paid at beginning of term

Annuity due $= \ddot{a}_{\overline{n}|} = 1 + V + \dots + V^{n-1} = \frac{1-V^n}{1-V} = \frac{1-V^n}{\frac{1+i}{1+i} - \frac{1}{1+i}} = \frac{1-V^n}{i/(1+i)} = \frac{1-V^n}{iV} = \boxed{\frac{1-V^n}{d}}$

Perpetuity due $= \ddot{a}_{\overline{\infty}|} = \frac{1}{d}$

another identity $\boxed{V\ddot{a}_{\overline{n}|} = a_{\overline{n}|}}$

$q_x \equiv \frac{d_x}{l_x}$ recall $d_x = l_x - l_{x+1}$ $\quad {}_k q_x$ If $k=1$, ${}_1q_x = q_x$

Ex. 4 from Ch 1

$$q_x \equiv \frac{d_x}{l_x} = \begin{cases} 4 \times 10^{-4} & 5 \le x < 30 \\ 8 \times 10^{-4} & 30 \le x \le 55 \end{cases}$$

Find $S(x)$, l_x, d_x, ... (if) $l_0 = 10^5$; $S(5) = 0.96$

Soln: $\frac{S(x+1)}{S(x)} = \frac{l_{x+1}}{l_x} = \frac{l_x - d_x}{l_x} = 1 - \frac{d_x}{l_x} = 1 - q_x$

$$\Leftrightarrow l_x \underset{\text{defn.}}{=} l_0 \times S(x) = (1-q_0)(1-q_1)\dots(1-q_{x-1})$$

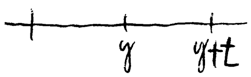
$$\Rightarrow \frac{S(x)}{S(5)} = (1-.0004)^{x-5} \quad \text{for } x = 5, 6, \dots, 30$$

$$l_x = 96000 \times (0.9996)^{x-5} \qquad S(30) = 0.940446$$

$x = 31-55$

$$\frac{S(x)}{S(30)} = (1-.0008)^{x-5}$$

$$\frac{S(y) - S(y-t)}{S(y)}$$



$S(y) = \int_y^\infty f(t)\,dt$; $f(t) = -S'(t)$ $\qquad T \equiv$ age of death; $Pr(T \ge y) = S(y)$

$$\rightarrow \lim_{\varepsilon \to 0^+} \frac{Pr(y \le T \le y+\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_y^{y+\varepsilon} f(t)\,dt = f(y)$$

Defn. ${}_tP_y \equiv P(T \ge y+t \mid T \ge y) = \frac{S(y+t)}{S(y)}$ $\qquad {}_tP_y = \frac{S(y+t)}{S(y)}$

${}_tq_y = 1 - {}_tP_y$ $\qquad {}_tq_y = \frac{S(y) - S(y+t)}{S(y)}$ Age-specific death rate (p. 48 of text)

Defn. Limiting Death rate (per unit time as $\varepsilon \downarrow 0$)

$$\frac{{}_\varepsilon q_x}{\varepsilon} \equiv \boxed{\text{force of mortality}}$$

{ failure intensity
{ hazard " (rate)

$${}_\varepsilon q_x = \frac{1}{\varepsilon S(y)} \int_y^{y+\varepsilon} f(t)\,dt \underset{\varepsilon \downarrow 0}{\longrightarrow} \frac{f(y)}{S(y)} \Rightarrow \underset{\mu_y}{\mu(y)} = \frac{f(y)}{S(y)} = -\frac{S'(y)}{S(y)} = -\frac{d}{dy}(\ln(S(y)))$$

So, $\int_0^y \mu(t)\,dt = -\ln(S(t))\Big|_0^y = -\ln(S(y))$ AND $\int_y^{y+t} \mu(s)\,ds = \ln(S(y)) - \ln(S(y+t))$

$$\Leftrightarrow S(y) = e^{-\int_0^y \mu(t)\,dt}$$

$$\hookrightarrow {}_tP_y = \frac{S(y+t)}{S(y)} = e^{-\int_y^{y+t} \mu(s)\,ds}$$

Homework Problem:

You are given

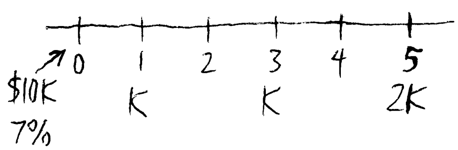
i) ${}_3p_{70} = 0.95$

ii) ${}_2p_{71} = 0.96$

iii) $\int_{71}^{75} \mu(x)\,dx = 0.107$

Calculate ${}_5p_{70}$. (Hint: ${}_3p_{70} = p_{70} \times {}_2p_{71}$)

Problem 1.12 from Slud:



Soln: $\$10{,}000 = K(v + v^3 + 2v^5)$ where $v = \frac{1}{1+i}$

Problem 1.10 from Slud for HW

Outline of the course

Theory of Interest	Life-Table Stat & Survival	Contingent Contracts
Amortization/Mortgage	Cohort-Life Table	⊕ Life Annuity
Annuities	$l_x, d_x, q_x,$ —	Insurance
Imm. Ann.	Survival Function	Term/Whole Life
Ann. Due	${}_kp_x, \mu(x),$ —	Pure Endorsement
Perpetuity	Interpolation	Endorsement Insurance
Incr/Decr	$\boxed{\text{Models}}$	Net Single Premiums
		Premiums
		Equation-of-value
		Reserves/Loss

HW 2: 1.10, 1.12, 2.2, 2.3 (from Slud) plus H.1

Problem H.1

Given i) ${}_3p_{70} = 0.95$ Calculate ${}_5p_{70}$

ii) ${}_2p_{71} = 0.96$

iii) $\int_{71}^{75} \mu_x \, dx = 0.107$

Slud 2.2

$$f(t) = \begin{cases} \frac{1}{360}(1 + t/10), & 20 \le t \le 80 \\ 0, & \text{otherwise} \end{cases}$$

Suppose (20) has random lifetime Z

life aged 20

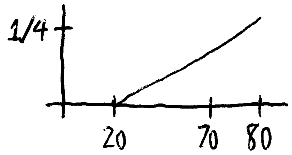
a) Pay $10^6 * (1.4 - Z/50)$ @ exact age of death $\boxed{\text{if}}$ this occurs between 20 & 70
expected payment

b) PV, nominal rate of interest is $(e^{0.08} - 1)$/yr

0. Is $f(t)$ a density?

$$\int_{20}^{80}(1+t/10)\,dt = \left(t+t^2/20\right)\Big|_{20}^{80} = (80-20)+\frac{1}{20}(80^2-20^2) = 60+\frac{1}{20}(6000) = 60+300=360 \checkmark$$

For any $g(T)$, $E[g(T)] = \int_0^w g(t)f(t)\,dt$



$S(0)=1$

$S(20)=1$

a) $g(t)=10^6\times(1.4-t/20)$

$$E[g(t)] = \int_{20}^{70}\left(10^6\times(1.4-t/50)\right)\left(\frac{1}{360}(1+t/10)\right)dt = \$324.\_\_\_$$

$$=\int_0^w g(t)f(t) = \int_0^{20}+\int_{20}^{70}+\int_{70}^{80} =$$

b) Recall: $\delta=\ln(1+i)=\lim i^{(m)}$

$$\left(1+i^{(m)}/m\right)^m = 1+i_{eff}$$

$$e^{\ln(1+i)} = 1+i \Rightarrow i=e^{\ln(1+i)}-1 = i_{eff}$$

nominal rate of interest $=(e^{0.08}-1)/yr$

instantaneous rate of interest $\leftrightarrow$ force of interest

$g(t)=e^{-.08(t-20)}g_1(t)$ where $g_1(t)$ is $g(t)$ from part (a)

$\sum P(t) * Pr(\text{Death @ } t) * V^t * \Delta t$

$\int_0^\infty g(t)f(t)\,dt$

Slud 2.3 Cont. r.v. T has hazard rate

$$h(t) = \begin{cases} 10^{-3} * [7.0 - 0.5t + 2e^{t/20}] & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Construct a cohort life table with h(t) given; based on integer ages up to 70 & radix $\equiv l_0 = 10^5$

Note: do calculation for 10, 20, 30, ..., 70

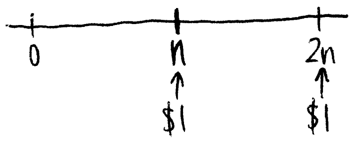
$$\begin{cases} S(x) = e^{-\int_0^x h(t)\,dt} \\ l_x = S(x) * l_0 \end{cases}$$

	l_x	
10:	93,150.95	0.932

b) $\dfrac{S(30)}{S(3)}$ use conditional prob.

HW Comments

Slud 2.1



$PV(\$1 @ t=n + \$1 @ t=2n) = \$1$

$1 = V^n + V^{2n} \Rightarrow$ quadratic eqn.

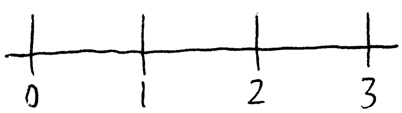
$w = V^n$

$w^2 + w - 1 = 0$

Slud 2.7

$n(t) = 0.01t \;;\; 0 \le t \le 3$

Find single effective rate of interest



$\$1 \to \$1 * (1 + i_{eff})$

$(1 + i_{eff})^3 = e^{\int_0^3 \ln(\;)}$

Problem H.2

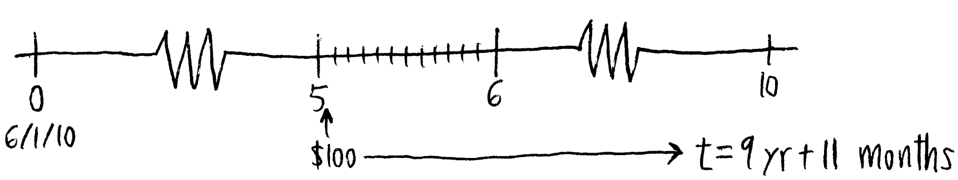
Time 0 = June 1, 2010

Buy a deferred annuity which pays \$100/month for 60 months beginning on June 1, 2015. $i_{eff} = 6\%$

Calculate the cost

Step 1: Find $i^{(12)}$

Soln: PValue =
(PV)



2 choices:

$\begin{cases} \text{① Imm. annuity starting @ } t = 4 \text{ yr } 11 \text{ months} \\ \text{② Annuity due @ } t = 5 \text{ yrs} \end{cases}$

$\underset{\substack{\uparrow \\ 100 \times m}}{1200} * V^5 * \ddot{a}^{(12)}_{\overline{5|}} \to \dfrac{1 - V^5}{d^{(12)}}$

$100 \times (V^5 + V^{61/12} + V^{62/12} + \ldots + V^{119/12}) = 100 \times V^5 \times (1 + V^{1/12} + \ldots)$

Problem H.3

$$ {}_tP_0 = \begin{cases} 1 - t/110, & 0 \le t \le 110 \\ 0, & \text{otherwise} \end{cases} \qquad {}_tP_0 = \frac{S(0+t)}{S(0)} $$

40^{th} birthday is 6/1/10

$i_{eff} = 5\%$

funds a gift to charity payable on 6/1/35

$$ \begin{cases} \text{If alive, } \$100000 \\ \text{otherwise, } \$25000 \end{cases} $$

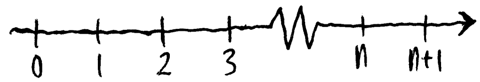
PV(gift) = Prob (alive) * PV (100000) + Prob (not alive) * PV (25000)

Perpetuities

$a_{\overline{\infty}|} \rightarrow a_{\overline{\infty}|} \qquad a_{\overline{\infty}|} \neq a_{\overline{\infty}|}$

A) Imm. Perpetuity pays \$1 @ $t = 1, 2, 3, \ldots$

$(m=1)$



$$a_{\overline{n}|} = v + v^2 + \ldots + v^n = v(1 + \ldots + v^{n-1})$$

$$= \frac{1-v^n}{i} \qquad = v\left(\frac{1-v^n}{1-v}\right)$$

$$v \equiv \frac{1}{1+i} \qquad \frac{v}{1-v} = \frac{v}{\frac{1+i}{1+i} - \frac{1}{1+i}} = \frac{v}{\frac{i}{1+i}} = \frac{1}{i}$$

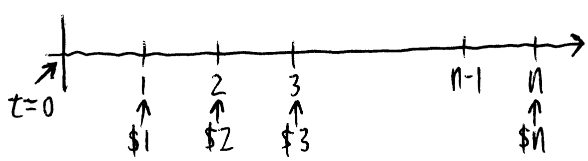
$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} \qquad \text{Take } \lim_{n\to\infty}: \ a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

Fact: $v = \frac{1}{1+i} < 1$

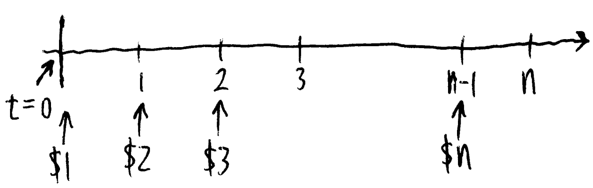
B) Increasing Annuity

1) Immediate Incr Annuity, $(Ia)_{\overline{n}|}$

2) Incr Annuity Due, $(I\ddot{a})_{\overline{n}|}$

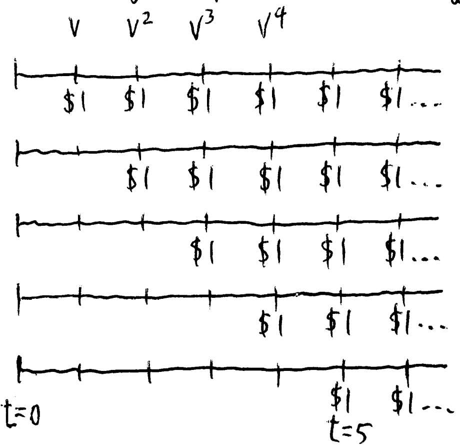


$(Ia)_{\overline{n}|}$



$(I\ddot{a})_{\overline{n}|}$

Increasing Perpetuities $(Ia)_{\overline{\infty}|}$, $(I\ddot{a})_{\overline{\infty}|}$



$$a_{\overline{\infty}|} + v a_{\overline{\infty}|} + v^2 a_{\overline{\infty}|} + v^3 a_{\overline{\infty}|} + v^4 a_{\overline{\infty}|} + v^5 a_{\overline{\infty}|} + \ldots = (Ia)_{\overline{\infty}|}$$

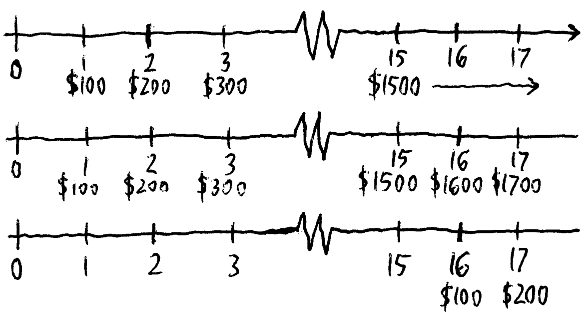
$$(Ia)_{\overline{\infty}|} = a_{\overline{\infty}|}(1+v+v^2+\ldots) = a_{\overline{\infty}|}\left(\lim_{n\to\infty} \frac{1-v^n}{1-v}\right)$$

$$= a_{\overline{\infty}|}\left(\frac{1}{1-v}\right) = \frac{1}{i} \times \frac{1}{d} = \frac{1}{id}$$

Note: $\frac{1}{1-v} = \left(\frac{1+i}{1+i} - \frac{1}{1+i}\right)^{-1} = \left(\frac{i}{1+i}\right)^{-1} = \frac{1}{iv} = \frac{1}{d}$

Old Exam Problem

Find the PV of a perpetuity which pays \$100 @ $t=1$ yr, \$200 @ $t=2$ yr, ..., \$1500 @ $t=16$ yr onward



$100(Ia)_{\overline{15}|} + v^{15} * 1500 * a_{\overline{\infty}|}$

$\$100(Ia)_{\overline{\infty}|}$

$\$100\, v^{15}(Ia)_{\overline{\infty}|}$

$$X = \$100(Ia)_{\overline{\infty}|} - \$100 v^{15}(Ia)_{\overline{\infty}|} = \$100(Ia)_{\overline{\infty}|}(1-v^{15})$$

HW Tips

2.1. Quadratic in v^n

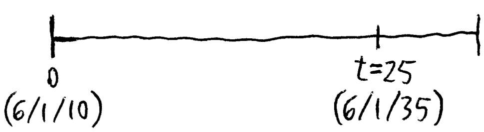
2.7. $i_{eff} = i = 6\%$ $\quad i \leftrightarrow i^{(m)} \quad a_{\overline{n}|} \leftrightarrow a^{(m)}_{\overline{n}|}$

$$PV = \$100 v^5 (1 + v^{1/12} + v^{2/12} + \ldots + v^{59/12})$$

$$a^{(m)}_{\overline{n}|} = v^{1/m}\, \ddot{a}^{(m)}_{\overline{n}|}$$

Lara Problem.

Lara is age 40



Alive → \$100,000
Not Alive → \$25,000

Calculate EPV = expected present value

2 outcomes

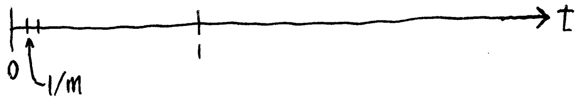
A: Alive @ 6/1/35 ⟷ Prob = p
B: Not Alive @ 6/1/35 ⟷ Prob = q $\qquad p+q=1$

T_x or $T(x) = E(T-x \mid T \geq x)$ (for Cohort Life Table Probability)

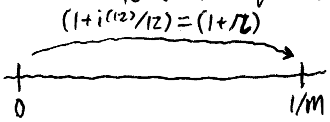
Mortgages & Loans

$L \equiv$ loan   Most (formal) calculations are in terms of per \$1

Fact: Mortgages are paid monthly ($m=12$) in arrears



Interest rate (APR) quoted



Assume $i \equiv i_{eff}$

Recall $1+i=\left(1+\frac{i^{(12)}}{12}\right)^{12} \rightarrow i^{(12)}/12$

$X=$ <u>level</u> payment

Amortization

1 month: $L \rightarrow L(1+r)-X$

2 months: $L \rightarrow (L(1+r)-X)(1+r)-X = L(1+r)^2-X(1+(1+r))$

3 months: $L \rightarrow (L(1+r)^2-X(1+(1+r)))(1+r)-X = L(1+r)^3-X(1+(1+r)+(1+r)^2)$

k months: $L \rightarrow L(1+r)^k - X\left(\sum_{j=0}^{k-1}(1+r)^j\right)$

@ $k=360$: $L \rightarrow 0 = L(1+r)^{360} - X\left(\sum_{j=0}^{359}(1+r)^j\right)$

$$X=\frac{L(1+r)^{360}}{\sum_{j=0}^{359}(1+r)^j}=\frac{rL(1+r)^{360}}{(1+r)^{360}-1}$$

where $\sum_{j=0}^{359}(1+r)^j=\frac{1-(1+r)^{360}}{1-(1+r)}=\frac{1-(1+r)^{360}}{-r}=\frac{(1+r)^{360}-1}{r}$

$$(1+r)^{360}=\left(\left(1+\frac{i^{(12)}}{12}\right)^{12}\right)^{30}=(1+i)^{30}$$

\$X/month $\longleftrightarrow$ \$12X/year

$12Xa_{\overline{30|}}^{(12)}$

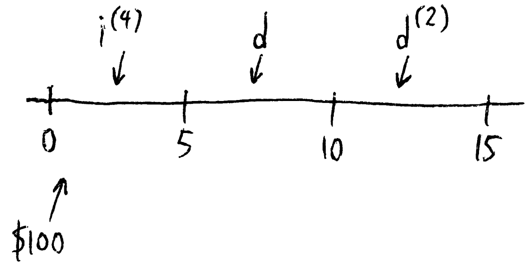
$mXa_{\overline{n|}}^{(m)}$ recall, $a_{\overline{n|}}^{(m)}=\frac{1-v^n}{i^{(m)}}$

Zero-coupon

OID = original issue discount

Review for Exam 1

Slud 2.8.



$$\$100 \times \left(1+\frac{i^{(4)}}{4}\right)^{4\times 5} \cdot (1-d)^{-5}\left(1-\frac{d^{(2)}}{2}\right)^{-2\times 5}$$

$$1+i=\frac{1}{1-d}$$

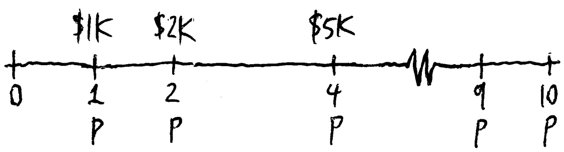
$$1+\frac{i^{(2)}}{2}=\frac{1}{1-\frac{d^{(2)}}{2}}$$

HW 4 Hints / Tips

Prob 1: $\delta = 2$, Compute $i, i^{(4)}$

$i =$ ____ (some function of δ) $\quad \ln(1+i)=\delta$

Prob 2:



$i = 10\%$

part a: $1000v + 2000v^2 + \dots = Pa_{\overline{10|}}$

part b: $1000(v + 2v^2 + 5v^4) = 7000v^t$

$\mathring{e}_x \equiv$ complete expectation of life (for (x))

$e_x =$ curtate expectation = expected # of whole years of remaining life

$[T] \equiv K$

Prob 3: $S(x) = 1 - x^2/36$, compute p_x & μ_x

$$p_x = \frac{S(x+1)}{S(x)}$$

$x=3$

\# of whole yrs remaining in life

$P_r([T-x]=0) = q_3$

$P_r([T-x]=1) = p_3 q_4$

Prob 4:

No problems involving integration by parts on exam

$$E[T] = \int_0^{\omega} t f(t)\,dt = -tS(t)\Big|_0^{\omega} + \int_0^{\omega} S(t)\,dt = \int_0^{\omega} S(t)\,dt$$ useful identity

Problem similar to HW 2 Extra Problem will be on exam

For the exam:

- concept of independent lives
- survival functions

⋮

HW 3 Comments:

$S(x) \neq 98 - x$

$S(x) = \dfrac{98-x}{98}$

Be able to do Problem 6

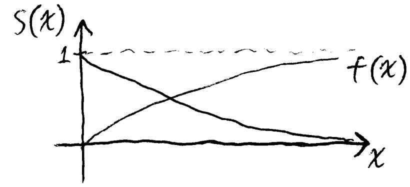
Problem 7 & 8 were on a previous exam

5/23/10

$S(x)$ = survival function

Defn: $f(x) = -S'(x)$

define $F(x) = 1 - S(x)$



$S(0) = 1$ (100% survival) $\quad$ $f(x)$ = failure rate

$$\mu(x) = \frac{-S'(x)}{S(x)} = \frac{f(x)}{S(x)}$$ force of mortality (act on those who remain)

$\frac{1}{\text{time}}$, $(\text{time})^{-1}$

Exponential

$\mu \equiv$ constant

$$S(t) = e^{-\int_0^t \mu(s)\,ds} = (\mu(t) = \text{constant} = \mu) = e^{-\int_0^t \mu\,ds} = e^{-\mu t}$$

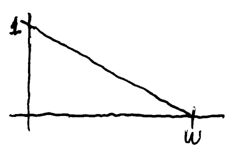
Gompertz-Makeham

$$\mu(y) = \underset{\substack{\uparrow \\ \text{accidental death}}}{A} + \underbrace{Bc^y}_{\text{Gompertz}} \qquad A, B, c > 0 \;;\; c \geq 1$$

$$c^y \equiv e^{y \ln c}$$

DeMoivre

$$S(x) = \frac{\omega - x}{\omega}$$



Acturial Notation

$$a_{\overline{n|}} = v + v^2 + v^3 + \dots + v^n = \frac{1 - v^n}{i}$$

$$\ddot{a}_{\overline{n|}} = 1 + v + v^2 + \dots + v^{n-1} = \frac{1 - v^n}{d} = \frac{1 - v^n}{1 - v}$$

$$1 - v = 1 - \frac{1}{1+i} = \frac{(1+i) - 1}{1+i} = \frac{i}{1+i} = iv = d \qquad (1-d) = \frac{1}{1+i} = v \Rightarrow d = 1 - v$$

Annuities certain

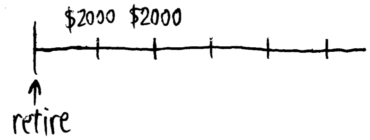
$$a_{\overline{n|}}^{(m)} = \frac{1 - v^n}{i^{(m)}}, \quad \ddot{a}_{\overline{n|}}^{(m)} = \frac{1 - v^n}{d^{(m)}}$$

Perpetuities

$a_{\overline{\infty|}}$; $\ddot{a}_{\overline{\infty|}}$

$(Ia)_{\overline{n|}}$; $(I\ddot{a})_{\overline{n|}}$ \\ decreasing annuities

life annuities



3% arithmetic decrease

$a_x^{(m)} + 0.03(Ia)_x^{(m)}$

$a_{\overline{x:n|}}^{(m)}$ if $n \neq 1$

$a_{\overline{x:n|}}^{(m)}$

$\ddot{a}_{\overline{x:n|}}$

$\bar{a}_x$ $\bar{A}_x$

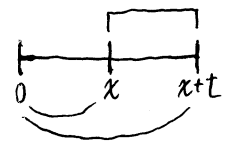
$\bar{a}_{\overline{x:n|}}$ $\bar{A}_{\overline{x:n|}}$

Proposition

$$\frac{\partial}{\partial x}({}_tp_x) = {}_tp_x * (\mu_x - \mu_{x+t})$$

$$\mu(x) = \frac{-d}{dx}(\ln(S(x))) = \frac{-S'(x)}{S(x)} = \frac{f(x)}{S(x)}$$

$${}_tp_x = \frac{S(x+t)}{S(x)} = \frac{e^{-\int_0^{x+t}\mu(s)\,ds}}{e^{-\int_0^{x}\mu(s)\,ds}} = e^{-\left(\int_0^{x+t}\mu(s)\,ds - \int_0^{x}\mu(s)\,ds\right)}$$



Proof of proposition

$$\frac{\partial}{\partial x}({}_tp_x) = \frac{\partial}{\partial x} e^{-\int_x^{x+t}\mu(s)\,ds} = {}_tp_x(-\mu(x+t) - \mu(x)) = {}_tp_x(\mu(x) - \mu(x+t))$$

$$\frac{\partial}{\partial x} e^{g(x)} = g'(x)e^{g(x)}, \quad \frac{\partial}{\partial x}\int_{g(x)}^{h(x)} \mu(s)\,ds = h'(x)\,\mu(h(x)) - g'(x)\,\mu(g(x))$$

$${}_tp_x = \frac{S(x+t)}{S(x)} \qquad \backslash\backslash \quad {}_tq_x = 1 - {}_tp_x \qquad p+q=1$$

$$\frac{\partial}{\partial x}({}_tp_x) = \frac{S'(x+t)S(x) - S'(x)S(x+t)}{(S(x))^2} = \frac{S'(x+t)}{S(x)} - \frac{S'(x)}{S(x)}\cdot\frac{S(x+t)}{S(x)}$$

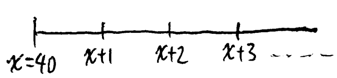
$$= \frac{S'(x+t)}{S(x+t)}\cdot\frac{S(x+t)}{S(x)} - \frac{S'(x)}{S(x)}\cdot\frac{S(x+t)}{S(x)} = {}_tp_x(\mu_x - \mu_{x+t})$$

curtate expectation of life $e_x \longrightarrow E([T-x])$

complete expectation of life $\mathring{e}_x$

$[y] \equiv$ greatest integer $\leq y$ $\qquad [3.9] = 3$

Curtate table



[probability mass function for value of $T-x$)

Tabulated (l_x, d_x) or $S(x)$

$\downarrow$

${}_tp_x$; $t = 0, 1, 2, \ldots$

${}_tq_x$

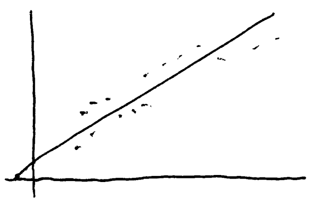
$$E[g(x)] = \sum g(x_i)\,p(x_i)$$

$\uparrow$

$Pr(X = x_i) \qquad p(x) \leftrightarrow f(x)$

$$E([T-x]) = 0 * Pr(T-x<1) + 1 * Pr(1 \le T-x < 2) + 2 * Pr(2 \le T-x < 3)$$
$$= 0*(1-p_x) + 1*(p_x - {}_2p_x) + 2*({}_2p_x - {}_3p_x) + \dots$$
$$= 0 * q_x + 1 * p_x(1-p_{x+1}) + 2 * {}_2p_x(1-p_{x+2}) + \dots$$
$$= 0 * q_x + p_x q_{x+1} + 2\, {}_2p_x\, q_{x+2}$$
$$= \sum_{k=0}^{\omega} k\; {}_kp_x\, q_{x+k} = \sum_{k=0}^{\omega} k({}_kp_x - {}_{k+1}p_x) = \sum_{k=x}^{\omega} g(t_k) * Pr(T = t_k)$$

$$c^x \equiv e^{x \ln c}$$



→ infer
least square

$${}_kp_x\, q_{x+k} = \sum_{k=0}$$

Plot model by Bc^x

$$\ln(Bc^x) = \ln B + \underbrace{\ln(c^x)}_{x \ln(c)}$$

$$Q(x) \sim \underset{\substack{\uparrow \\ \ln B}}{\beta_0} + \underset{\substack{\uparrow \\ \ln c^x}}{\beta_1} x$$

→ $\mu(x) = Bc^x$

→ $\mu(x) = A + Bc^x = A + Bc^x + D$

DeMoivre $\frac{w-x}{w}$

Exponential $S(x) \sim e^{-\mu x}$

Insurance (term or whole life)
Pure Endowment
Life Annuities

$A^{1}_{x:\overline{n|}}$ ~ Pays \$1 at the end of the year of death, provided that
$T < x+n$
$T - x < n$

<=> Actuarial PV
Expected PV (EPV)

$$Vq_x + V^2 p_x q_{x+1} + V^3\, {}_2p_x\, q_{x+2}$$

$A_x \equiv$ whole life

${}_tp_x$ as an estimator

view as a data life-table $l_0, l_1, l_2, l_3, \ldots$

a) Might arise from actual data

e.g. Acturial Publications

b) $S(x)$ given $\to l_0, l_1$

Given l_0, ${}_tp_x = \frac{S(x+t)}{S(x)}$

point of view of probability/statistics: ${}_tp_x$ is a population variable (not random)

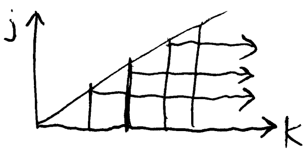
We have l_0 (or l_x) samples & we view $\frac{l_{x+t}}{l_x}$ as a r.v. which estimates ${}_tp_x$

Lemma 3.1

Let Z be nonnegative, integer value w, then

$E(Z) = \sum_{j=1}^{\infty} P(Z \geq j)$ $\qquad E(Z) \underset{\text{defn.}}{\equiv} \sum_j j\,Pr(Z=j)$ \\ Fubini-Tonnelli

Proof: $E(Z) = \sum_{k=0}^{\infty} k\,p_Z(k) = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} p_Z(k)\right) = \sum_{j=1}^{k}\left(\sum_{k=1}^{\infty} P(Z \geq j)\right) = \sum_{j=1}^{\infty} P(Z \geq j)$



e_x = curtate life expectancy

$= E([T] - x \mid T \geq x)$ $\qquad // \ x \in \mathbb{Z}$

$= \sum_{t=x}^{w-1} \frac{P(t \leq T < t+1)}{P(T \geq x)}(t-x) = \sum_{j=0}^{w-x-1} {}_jp_x * (1 - p_{x+j}) * j$

HW Tips

3.3. life aged 20 \\ 20

T ≡ exact age at death $\qquad f_T(t) = 0.02\,(t-20)\,e^{-(t-20)^2/100}$

$$\int_{20}^{\infty} f_T(t)\,dt = -e^{-(t-20)^2/100}\Big|_{20}^{\infty} = 1$$

a) $\$10^6\,(1-T/50)$ $\qquad // = g(T)_1$

b) discount this by factor $\exp(-0.08\,(T-20))$

$g(T)_2 = \$10^6\,(1-T/50)\,e^{-0.08\,(T-20)}$

$$\int_{20}^{70} g(t)\,f_T(t)\,dt \qquad c\int_{20}^{70} e^{-(t-20)^2/100}\,dt \qquad \tilde{c}\int_{20}^{70} e^{-z^2/2}\,dz$$

3.5. Gompertz-Makeham

a) Show ${}_tp_x = s^t g^{c^x}(c^t-1)$ where $s = e^{-A}$, $g = \exp(-B/\ln c)$, $c^x = e^{x\ln c}$

$$\frac{\partial}{\partial x} c^x = \ln c\; e^{x\ln c} = \ln c\,(c^x)$$

Proof: $\quad {}_tp_x = \dfrac{S(x+t)}{S(x)} = e^{-\int_x^{x+t}\mu(s)\,ds}$

b) Suppose ${}_{10}p_{50}$, ${}_{10}p_{60}$, ${}_{10}p_{70}$

Show $\quad c = \left(\dfrac{\log({}_{10}p_{70}) - \log({}_{10}p_{60})}{\log({}_{10}p_{60}) - \log({}_{10}p_{50})}\right)^{1/10}$

5. Obtain μ_x if $l_x = k s^x w^{x^2} g^{c^x}$

$$\mu_x = \frac{-S'(x)}{S(x)} = \frac{\left(\tfrac{d}{dx}s^x\right) w^{x^2} g^{c^x} + \ldots}{l_x}$$

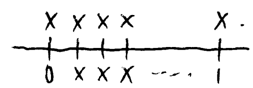
$$S(x) = \frac{l_x}{l_0} \qquad S(x) = \frac{k}{k_0}$$

$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$ later on...

HW Tips

3.8. Express in terms of $a_{\overline{n}|}^{(m)}$, the PV of an annuity

$100/month 1st yr
$200/month 2nd yr
⋮
$1000/month 10th yr



$\underline{\$1200\, a_{\overline{1}|}^{(12)} + \$2400\, v a_{\overline{1}|}^{(12)} + \dots} \longleftarrow$ not acceptable

$\rightarrow \$1200\left(a_{\overline{n}|}^{(m)} + v a_{\overline{n-1}|}^{(m)} + v^2 a_{\overline{n-2}|}^{(m)} + \dots\right)$ let $n=10$; $m=12$

3.11. Find EPV of 5% APR of an investment where proceeds

$10,000 @ $t=5$ w/ prob ½
$20,000 @ $t=10$ w/ prob ½

define Z = r.v. proceeds $Z \in \{v_1, v_2\}$

4.1. a) expected remaining life for age 20 (20)

b) $\dfrac{{}_{7/12}q_{40}}{q_{40}}$ more formal discussion next time

i) Weibull (0.00634, 1.2) $S(t) = e^{-0.00634t^{1.2}}$

ii) Lognormal $(\log(50), 0.325^2)$ $S(t) = 1 - \Phi(\log(t) - \log(50)/0.325)$

iii) Piecewise-exponential with

$$\mu_t = \begin{cases} 0.015 & 20 \le t \le 50 \\ 0.03 & t \ge 50 \end{cases}$$

$${}_tq_x = 1 - {}_tp_x = 1 - \frac{S(x+t)}{S(x)}$$

$$\frac{{}_{7/12}q_{40}}{q_{40}} = \frac{\dfrac{S(40) - S(40+7/12)}{S(40)}}{\dfrac{S(40) - S(41)}{S(40)}} = \frac{1 - e^{-0.015 * 0.58\overline{3}}}{1 - e^{-0.015}}$$

$0.58\overline{3} = 7/12$

$$S(t) = e^{-\int_0^t \mu(s)\,ds} = e^{-0.015t}$$

$\ddot{a}_{x:\overline{n}|}$

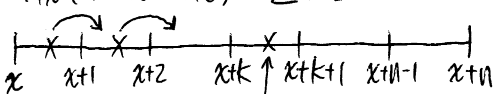
$a_{x:\overline{n}|}$

$A^{1}_{x:\overline{n}|}$

$A_{x:\overset{1}{\overline{n}|}} = {}_nE_x$

$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$

- Life annuity due $= 1 + vp_x + v^2\,{}_2p_x + \ldots + v^{19}\,{}_{19}p_x$
  $k(x) = k_x =$ (curtate) future life of x $\boxed{\text{r.v.}}$
  $e_x \equiv E(k_x)$ curtate expectation
- Immediate life annuity $= vp_x + v^2\,{}_2p_x + \ldots + v^{20}\,{}_{20}p_x$
  $a_{\overline{n}|} = v\,\ddot{a}_{\overline{n}|}$
- Insurance (term) $= vq_x + v^2 q_{x+1}\,p_x + \ldots + v^n q_{x+n-1}\,{}_{n-1}p_x$
  A_x (whole life) $= \sum \ldots$



$$\text{payment} = \begin{cases} 1 \text{ at } t = k(x)+1 & \text{if } k(x) < n \\ 0 & \text{otherwise} \end{cases}$$

$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$

$A_{x:\overset{1}{\overline{n}|}} = {}_nE_x$

pure endowment

$$\text{payment} \begin{cases} \$1 \ @\ t=n & \text{if} \\ \$0 & \text{otherwise} \end{cases} \rightsquigarrow \begin{cases} (x) \text{ is alive at } x+n \\ k(x) \geq 20 \end{cases}$$

$A_{x:\overset{1}{\overline{n}|}} = v^n\,{}_np_x$

$A_{x:\overline{n}|} = vq_x + v^2 p_x q_{x+1} + \ldots + v^n\,{}_{n-1}p_x\,q_{x+n-1}$

NSP = net single premium

HW Tips

3.5 $l_x = k s^x w^{x^2} g^{c^x}$

$$\mu_x = \frac{\ln s + 2(\ln w)x + (\ln g)(\ln c)c^x \, l_x}{l_x} = \ln s + 2(\ln w)x + (\ln g)(\ln c)c^x$$

Gompertz M-G: $\mu = Bc^x$

$\mu = A + Bc^x$

$\mu = A + Bc^x + Dx$

$$h(x) = \begin{cases} h_1(t) & 0 \le x \le A \\ h_2(t) & A \le x \le B \end{cases}$$

1. Weibull
2. lognormal
3. pw exp.

(ii) lognormal $(\log(50), 0.325^2)$

" (μ, σ^2)

$S(t) = 1 - \Phi\left((\log(t) - \log(50))/0.325\right)$ $\sigma = 0.325$, $\mu = \log(50)$

$S(t) = 1 - F(t)$ $F'(t) = f(t)$

$$E[T-20] = \frac{1}{S(20)} \int_{20}^{\infty} (t-20) f(t)\, dt$$

density from $t=0$

$$f(t) = -S'(t) = \frac{1}{S(20)} \int_{20}^{\infty} (t-20)\, dt$$

$$-S'(t) = \frac{1}{\sigma t} e^{-(\log t - \mu)^2/2\sigma^2} \qquad \Phi'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

4.1 $$E[T-20] = \frac{1}{S(20)} \int_{20}^{\infty} (t-20) f(t)\, dt$$

$$= \frac{1}{S(20)} \int_{20}^{\infty} \frac{t-20}{\sqrt{2\pi}\,\sigma t} e^{-\left[\frac{(\log t - \mu)^2}{2\sigma^2}\right]} dt = \frac{1}{S(20)} \int_{\frac{\log 20 - \mu}{\sigma}}^{\infty} e^{\mu + \sigma v} \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv - 20$$

$v = \frac{\log t - \mu}{\sigma}$ $dv = \frac{dt}{\sigma t}$ $t = e^{\mu + \sigma v}$

$$\int_{\frac{\log 20 - \mu}{\sigma}}^{\infty} e^{-v^2/2} dv = S(20) \Rightarrow e^{\sigma v} e^{-v^2/2} = e^{\sigma^2/2} e^{-(v-\sigma)^2/2} = e^{\frac{\sigma^2}{2} - \frac{(v^2 - 2\sigma v + \sigma^2)}{2}}$$

$$\frac{1}{S(20)} \int_{20}^{\infty} f(t)\, dt = 1$$

Final answer ≈ 32.795 (just to check if you did problem correct)

$\$1200\, a^{(12)}_{\overline{10}|}$ $\quad a^{(m)}_{\overline{n}|}$

$\$1200\, a^{(12)}_{\overline{9}|}\, v$ $\quad a^{(m)}_{\overline{n-1}|}$

<u>life table (m=1 case)</u>

NSP
net single
premium
by actuary

$$\left(\begin{array}{l} A^{1}_{x:\overline{n}|} \quad (= q_x v + p_x q_{x+1} v^2 + \dots + {}_{n-1}p_x\, q_{x+n-1} v^n) \\ A_{x:\overline{n}|}^{\;\;1} \equiv {}_nE_x \quad (= v^n\, {}_np_x) \\ A_{x:\overline{n}|} \\ A_x \end{array}\right.$$

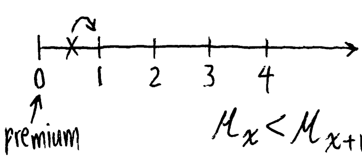
NSP <=> APV actuary present value

$\ddot{a}_{x:\overline{n}|}$; $\ddot{a}_x$

$a_{x:\overline{n}|}$; a_x $\longleftarrow$ (whole life) annuity

no one pays NSP in one payment

Level Premiums



$\mu_x < \mu_{x+1} < \mu_{x+2} < \dots$

$A^{1}_{50:\overline{30}|} = q_{50} v + p_{50} * A^{1}_{51:\overline{29}|}$ equivalence principle

$A^{1}_{x:\overline{n}|} = p + pv + pv^2 + pv^3 + \dots + pv^n = p\, \ddot{a}_{x:\overline{n}|}$

Go over quiz

$$ {}_kp_x = \frac{S(x+k)}{S(x)} = \frac{100-x-k}{100-x} \qquad {}_kq_x = 1 - {}_kp_x = \frac{(100-x)-(100-x-k)}{100-x} \Rightarrow q_x = \frac{1}{100-x} $$

$x=48$

$$ {}_kp_{48} = \frac{52-k}{52} \qquad {}_kq_{48} = \frac{1}{52} $$

a) $\ddot{a}_{48:\overline{2|}} = 1 + p_{48}V = 1 + \frac{51}{52}V = 1.94305$

b) $A^1_{48:\overline{2|}} = q_{48}V + p_{48}\,q_{49}V^2 = \frac{1}{52}V + \frac{51}{52}\times\frac{1}{51}V^2 = \frac{1}{52}(V+V^2)$

c) $A_{48:\overline{2|}} = A_{48:\overline{2|}}^{\ \ 1} + A^1_{48:\overline{2|}} = {}_2p_{48}V^2 + \frac{51}{52}V^2 = \frac{50}{52}V^2 + \frac{51}{52}V^2$

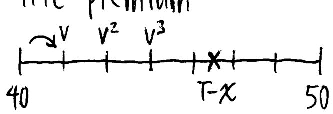
$$ \ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} V^k\,{}_kp_x $$

$x=40$ | $A_{40:\overline{10|}}$

Insured ↓ $\qquad$ $*1000$

k	$\ddot{a}_{40+k:\overline{10-k|}}$	${}_kV_{40:\overline{10|}}$	$A_{40+k:\overline{10-k|}}$	$A^1_{40+k:\overline{10-k|}}$	${}_kV^1_{40:\overline{10|}}$
0	7.84805	0	698.15	135.18	
1					
2					
3					
⋮					
8	1.9403		925.27	36.27	
9			961.54	18.35	

life premium



r.v. $Z \equiv V^{k+1}$

$\Pr(Z = V^{k+1}) = {}_kp_x\,q_{x+k} \qquad A_x = E[V^{k+1}] = \sum_{k=0}^{\infty} V^{k+1}\,{}_kp_x\,q_{x+k}$

level premium π

payment of P ($P_{40:\overline{10|}}$) @ $t=0,1,\dots,9$ if alive

premium to be paid: $PV = (1 + p_xV + {}_2p_xV^2 + \dots + {}_9p_xV^9)$

$PV(\text{premium}) = P\,\ddot{a}_{40:\overline{10|}}$

$PV(\text{policy}) = A_{40:\overline{10|}}$ $\qquad$ By eq. principal, $P = \dfrac{A_{40:\overline{10|}}}{\ddot{a}_{40:\overline{10|}}}$

4/13/10

$y = 1950, 1951$

A life-table for each y

$\mu_Y(t) = Ak(Y) + Bc^t$

$p_t^{Y+1} = (1+r)\, p_t^Y$

$k(1950) = 1$ find $K(Y)$

Gompertz-Makeham $\mu(Y) = A + Bc^Y$

$p_x^{Y+1} = (1+r)\, p_x^Y \iff e^{-\int_x^{x+1} \mu_{y+1}} = (1+r) e^{-\int_x^{x+1} \mu_y} \leftarrow k(y)$

Smokers and Nonsmokers

$\mu(t) = A + Bc^t$

$\mu^*(t) = A * Bc^t$

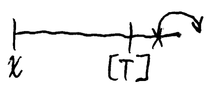
$$\begin{pmatrix} 6 & \frac{c^6-1}{\ln c} \\ 6 & c \cdot \frac{c^6-1}{\ln c} \end{pmatrix} \begin{pmatrix} A^* \\ B^* \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

solve this matrix

$$E[T-20] = \frac{1}{S(20)} \int_{20}^{\infty} (t-20) f(t)\, dt$$

$$E(g(T) \mid T \geq 20) = \frac{1}{S(20)} \int_{20}^{\infty} g(t) f(t)\, dt$$

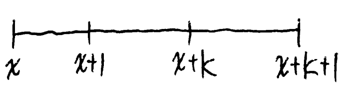
Ex. $A_x \rightarrow$ pays \$1 at end of the year of death



$A_x = E_x(v^{[T]+1-x}) = E(v^{T-x} \mid T \geq x)$

$A_x = q_x v + p_x q_{x+1} v^2$

$A_x^{(m)} \rightarrow$ payments \$1 @ end of 1/m yr



Take $\lim_{m \to \infty}$

$\bar{A}^1_{x:\overline{n}|} \rightarrow$ pays \$1 @ instance of death

$\bar{A}_x \rightarrow$ whole life version $\mu(x) \equiv \mu$

Theorem:

time varying instaneous force of interest

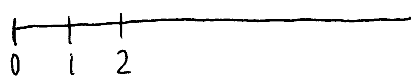
$r(t) = \alpha - \beta t$

$r(t) = 0 \leq t \leq T$

Question: find effective rate of interest on $[0,T]$

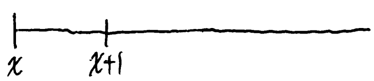
deposit @ $t=0 * (1+i)^T$

deposit $(1) * (1+i)^{T-1}$



$$\sum_{k=0}^{T-1} \text{deposit}(t=k)(1+i)^k = \text{accumulation}$$

Level premium



$A_x = v^k$

$k = [T-x]$

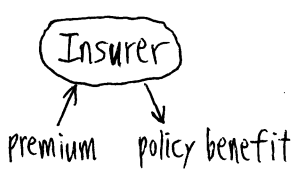
$A_x = E[v^{k+1}]$

↳ net single premium or net single risk premium

$A_x \longleftrightarrow P_x$

$A^1_{x:\overline{n}|} \longleftrightarrow P^1_{x:\overline{n}|}$

Policy Risk



Equivalence Principle

$E(\text{policy benefit}) - E(\text{premium}) = 0$

$P\ddot{a}_x = \text{present value}$

$$P_x = \frac{A_x}{\ddot{a}_x} \quad ; \quad P^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

7/20/10

Chapter 4 material

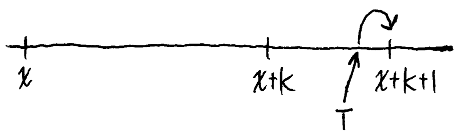
$A^1_{x:\overline{n|}}$, $A_{\overline{x:n|}}$, $\ddot{a}_{x:\overline{n|}}$

$\bar{A}_x$, $\bar{A}_{x:\overline{n|}}$, $\bar{a}_x$

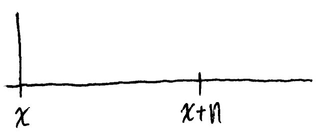
$P([T]=x+k \mid T \geq x) = {}_kp_x - {}_{k+1}p_x = {}_kp_x\, q_{x+k}$

term insurance payment $F(k)$ @ $t=k+1$

$E(v^{[T]-x+1} * F([T]-x) * I[T \leq x+n] \mid T \geq x)$



$v^{n+1} * F(k)$



$IB \equiv \chi B(y) = \begin{cases} 1 \\ 0 \end{cases}$

$$= \sum_{k=0}^{n-1} F(k)\, v^{k+1}\, {}_kp_x\, q_{x+k}$$

$$E\left(\sum_{k=0}^{\min(n,[T]-x)} v^k F(k)\right) = \sum_{k=0}^{n} v^k F(k)\, {}_kp_x \qquad E[v^{[T]+1} \mid T \geq x]$$

$$A^1_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k} \quad \backslash\backslash \quad A_x$$

$T \in \mathbb{R}$

$[T] = k$

T = time of death

$T-x$ = remaining life

$[T]$ whole yr lived from time 0

$$\ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^k\, {}_kp_x$$

$$A_{x:\overline{n|}}^{\;\;1} = E(v^n * I[T-x \geq n]) = v^n * {}_np_x$$

$$A_{\overline{x:n|}} = \sum_{k=0}^{\infty} v^{\min(n,k+1)}\, {}_kp_x\, q_{x+k} = \sum_{k=0}^{n-1} v^{k+1}({}_kp_x - {}_{k+1}p_x) + v^n p_x$$

$$= v\, {}_0p_x + (v^2 - v)\, {}_1p_x + (v^3 - v^2)\, {}_2p_x + \ldots + (v^n - v^{n+1})\, {}_np_x = 1 + (v-1)\, {}_0p_x = 1 + \sum_{k=0}^{n-1} (v^{k+1} - v^k)\, {}_kp_x$$

$$A_{x:\overline{n|}} = 1 - (1-v) \sum_{k=0}^{n-1} v^k\, {}_kp_x = 1 - (1-v)\, \ddot{a}_{x:\overline{n|}} = 1 - d\, \ddot{a}_{x:\overline{n|}}$$

$$A_{x:\overline{n|}} = 1 - d\, \ddot{a}_{x:\overline{n|}} \qquad \ddot{a}_{x:\overline{n|}} = \frac{1 - A_{x:\overline{n|}}}{d}$$

take a fixed value of r.v. T,

$$\ddot{a}^{(m)}_{\min}(T_m - x + \tfrac{1}{m}, n) = \frac{1 - v^{\min(T_m - x + \frac{1}{m}, n)}}{d^{(m)}}$$

$$\rightarrow \ddot{a}^{(m)}_{x:\overline{n}|} = E\left(\frac{1 - v^{\min(T_m - x + \frac{1}{m}, n)}}{d^{(m)}}\right) = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$$

$$v^t = e^{(\ln v)t}$$

$$\delta = \ln(1+i) \longleftrightarrow e^{\delta} = 1+i \longleftrightarrow e^{-\delta} = \frac{1}{1+i} = v$$

$$-\delta = \ln v \quad \Rightarrow \quad \delta = -\ln v$$

Continuous life annuity

Def: continuous payments at the rate of \$1/yr with duration $= \min(T-x, n)$

$$E(g(T)) = E(g(T) \mid T \geq x) = \frac{1}{S(x)} \int_x^{\infty} g(T) f(x)\, dy = \int_0^{\infty} g(x+t)\, \mu(x+t)\, {}_tp_x\, dt$$

$$\left(\begin{array}{l} f(x) = -S'(x) \\ \int_x^{\infty} \frac{f(y)}{S(x)}\, dy = 1 \end{array}\right)$$

$$= \frac{1}{S(x)} \int_0^{\infty} g(x+t) f(x+t)\, dt \quad \backslash\backslash \quad \begin{array}{l} y = x+t \\ dy = dt \end{array}$$

$${}_tp_x = \frac{S(x+t)}{S(x)} \qquad f(x+t) = -S'(x+t) \qquad \mu(x+t) = \frac{-S'(x+t)}{S(x+t)}$$

$$\rightarrow \frac{-S'(x+t)}{S(x)} = \frac{-S'(x+t)}{S(x+t)} \cdot \frac{S(x+t)}{S(x)} = \mu(x+t)\, {}_tp_x$$

$$g(y) = y - x \rightarrow \mathring{e}_x = E\{T - x\} = \int_0^{\infty} t\, \mu(x+t)\, {}_tp_x\, dt$$

$$g(y) = v^{y-x} \rightarrow \bar{A}_x = E\{v^{T-x}\} = \int_0^{\infty} v^t \mu(x+t)\, {}_tp_x\, dt$$

$$g(y) = v^{y-x} * I[y - x \leq n] \rightarrow A^1_{x:\overline{n}|} = E\{v^{T-x} I[T - x \leq n]\} = \int_0^{n} v^t \mu(x+t)\, {}_tp_x\, dt$$

$g(y) = y - x$

v^{y-x}

$v^{y-x} \cdot I\{y-x \le n\}$

$m=1$: A_x ; $A^1_{x:\overline{n|}}$; $A_{x:\overline{n|}}$ general m: $A_x^{(m)}$ ....

$a_{x:\overline{n|}}$ $\ddot{a}_{x:\overline{n|}}$

limit $m \to \infty$

$\bar{A}_x$; $\bar{A}^1_{x:\overline{n|}}$; $\bar{A}_{x:\overline{n|}}$

$\ddot{a}_{x:\overline{n|}}$

random variable

T = time of death

$T - x$

$$\begin{cases} k = [T] \\ k_x = [T-x] \end{cases}$$

$A_x = E(v^{[T-x]+1}) = vq_x + v^2 p_x q_{x+1}$

$A^1_{x:\overline{n|}} = E(v^{[T-x]+1})$

$A_{x:\overline{n|}} = E(v^{\min\{[T-x]+1, n\}})$

$A_{x:\overset{1}{\overline{n|}}} = \ldots = v^n {}_n p_x$

$A^1_{x:\overline{n|}} = A_{x:\overline{n|}} - A_{x:\overset{1}{\overline{n|}}}$

$\mathring{e}_x = E[T-x \mid T \ge x] = \int_0^\infty s\mu(x+s)\,{}_sp_x\,dx$ $\qquad$ $\bar{A}_x = E[v^{T-x} \mid T \ge x] = \int_0^\infty v^s \mu(x+s)\,{}_sp_x\,ds$

$$\mathring{e}_x = \frac{1}{S(x)} \int_x^\infty (t-x) f(t)\,dt \underset{t=s+x}{=} \frac{1}{S(x)} \int_0^\infty s f(s+x)\,ds = \frac{1}{S(x)} \int_0^\infty s(-S'(x+s)) \frac{S(x+s)}{S(x+s)}\,ds$$

$$= \int_0^\infty s \left(\frac{-S'(x+s)}{S(x+s)}\right)\left(\frac{S(x+s)}{S(x)}\right) ds = \int s\mu(x+s)\,{}_sp_x\,ds$$

$\mathring{e}_x = E[T-x \mid T \ge x] = \int_0^\infty s\mu(s+x)\,{}_sp_x\,ds$

$\bar{A}_x = E[v^{T-x} \mid T \ge x] = \int_0^\infty v^s \mu(x+s)\,{}_sp_x\,ds$

$\bar{A}^1_{x:\overline{n|}} = E[v^{T-x} \mid T \ge x] = \int_0^n v^s \mu(x+s)\,{}_sp_x\,ds$

$\bar{a}_x = \int^{\omega - x} v^y\,{}_yp_x\,dy$

$\bar{a}_{x:\overline{n|}} = \int_0^n v^y\,{}_yp_x\,dy$

Force of interest δ
Force of mortality μ

$V = \frac{1}{1+i}$ $\quad$ \$1 at 0, \$(1+i) at 1

$\delta = \ln(1+i) \quad e^{\delta} = (1+i)$

$S(x) = e^{-\int_0^x \mu(s)\,ds} = e^{-\mu x}$

${}_tp_x = e^{-\int_x^{x+t} \mu\, ds} = e^{-\{\mu s|_x^{x+t}\}} = e^{-\mu\{(x+t)-x\}} = e^{-\mu t}$

$\bar{A}_x = \int_0^{\infty} v^s \mu(x+s)\, {}_sp_x\, ds = \int_0^{\infty} e^{-\delta s} \mu e^{-\mu s}\, ds$

$= \mu \int_0^{\infty} e^{-(\delta+\mu)s}\, ds = \frac{\mu}{\delta+\mu}\left[-e^{-(\delta+\mu)s}\Big|_0^{\infty}\right] = \frac{\mu}{\delta+\mu}$

$\delta = \ln(1+i)$
$e^{\delta} = 1+i$
$e^{-\delta} = v$

Premium
insured pays premium
continuously
net premium (level)
Actuarial PV (APV) of policy benefit
APV of future premium = U

m=1 $\quad$ P P P P
general m $\quad$ x $\quad t=0 \rightarrow$

$L = v^T - P(\bar{A}_x)\,\bar{a}_{\overline{T}|}$

Prop. assuming net premium ($E(L)=0$)
then,

$$P(\bar{A}_{x:\overline{n}|}) = \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta \qquad n = \frac{1}{\bar{a}_x} - \delta$$

$\bar{A}_x = \frac{\mu}{\mu+\delta}$; $\bar{A}_{x:\overline{n}|}$

Prop.
$$\mathrm{Prob}\{[v^T - P(\bar{A}_x)\,\bar{a}_{\overline{T}|}] < 0\} = \left(\frac{\mu}{\mu+\delta}\right)^{\mu/\delta}$$

NSP = net single premium

Any (life contingent) contract

A_x ; $A^1_{x:\overline{n|}}$; $A_{x:\overset{1}{\overline{n|}}}$; $A_{x:\overline{n|}}$

$A^{(m)}_x$ . . . .

$\lim_{m\to\infty}$ $\bar{A}_x$

a_x ; $\ddot{a}_x$; $\bar{a}_x$

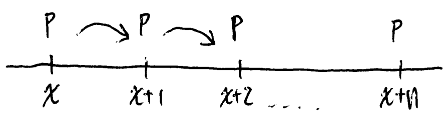
NSP ≡ actuarial PV $\qquad \bar{A}_x = E[V^{T-x} \mid T \geq x]$

Contract on x pays

i) \$1000 if x dies between x & $x+5$

ii) \$2000 " " " " $x+5$ & $x+10$

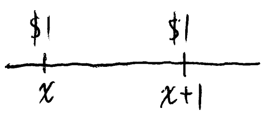
$$1000 A^1_{x:\overline{5|}} + 2000 A^2_{x:\overline{10|}} V^5 {}_5p_x = 1000 A^1_{x:\overline{10|}} + 1000 {}_5p_x V^5 A^1_{x+5:\overline{5|}}$$



(Standardized) level premium

$A^1_{x:\overline{n|}} \xrightarrow[\text{premium}]{\text{level}} P^1_{x:\overline{n|}} = P(A^1_{x:\overline{n|}})$

There are n premiums paid at $t = 0, 1, \ldots, n-1$

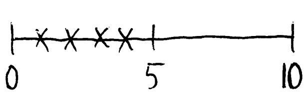


Equivalence Principle

$E[L] = 0$

$A^1_{x:\overline{n|}} = \underset{\substack{\uparrow \\ \text{net level premium}}}{P}(A^1_{x:\overline{n|}})\, \ddot{a}_{x:\overline{n|}} \qquad P = \dfrac{A^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$

Ex. $n = 10$, 5 level premiums



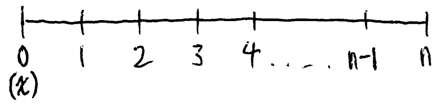
$$P = \frac{A^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{5|}}}$$

(x): Term insurance, duration $= n$ with 2 benefits

a) face amount, \$F: end of yr of death if $T-x \le n$

b) if x is alive after n yrs, the premiums are returned w/o interest

What is level premium?



$$F \cdot A^1_{x:\overline{n}|} + (nP) A_{x:\overline{n}|}$$

$$(nP)\, v^n {}_np_x$$

APV of the premium

$$P \ddot{a}_{x:\overline{n}|} = F \cdot A^1_{x:\overline{n}|} + nP A_{x:\overset{1}{\overline{n}|}}$$

Return of NSP

a. $A^1_{x:\overline{n}|}$ \$F

b. NSP @ $t=n$

Let $x =$ NSP of the contract

$$x = F \cdot A^1_{x:\overline{n}|} + x A_{x:\overset{1}{\overline{n}|}} \qquad P = \frac{x}{\ddot{a}_{x:\overline{n}|}}$$

Deferred 2 yrs; duration 3 yrs

$$v^2 {}_2p_x A^1_{x+2:\overline{3}|}$$

$$A_{x:\overset{1}{\overline{n}|}} \times A^1_{x+2:\overline{3}|}$$

$q_{40} = 0.1$ $\quad$ $l_{40} = 100000$

$q_{41} = 0.1$ $\quad$ $l_{41} = 90000$

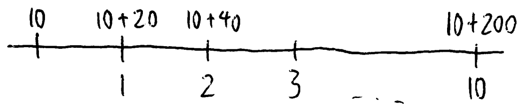
$q_{42} = 0.1$ $\quad$ $l_{42} = 81000$

$r(t)$ given force of interest

$r(t) = 0.08 - 0.02t \quad 0 \leq t \leq 10$

δ force of interest (constant)

$e^{\delta} - 1 = i_{eff} \Rightarrow e^{\delta} = i+1 \Rightarrow \delta = \ln(i+1)$



Q: Find i_{eff} for $0 \leq t \leq 10$

$A(10)$ = accumulation at $t = 10$ of all deposits

$$A(t) = 10e^{\int_0^{10} r(t)\,dt}$$

$$e^{-\delta} = \frac{1}{1+i} = v$$

$$A(t) = 10e^{\int_0^{10} r(t)\,dt} + 30e^{\int_1^{10} r(t)\,dt} + \ldots$$

$$= 10(1+i)^{10} + 30(1+i)^{9} + \ldots$$

let $w = 1+i$

$P(w) = A(t)$

$f(w) = 0$

$P(w) - A(10) = 0$