STAT 400

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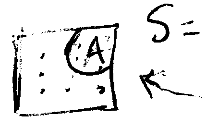
$\mathcal{E}$: (statistical experiment) = procedure with outcomes i S

S : the sample space



$S = \{s_1, s_2, s_3, \ldots\}$

events are denoted by capital letters

A occurs when the outcome is one of the pt in A

"A occured" $\Leftrightarrow$ $s \in A$ (s being my elementary even

$\emptyset \Leftrightarrow$ never occurs.

$n = card(S) = |S| = \#$ of elmts in S

# of events U can construct is 2^n. (0 or 1 each time)

A, B - events

$C = A \cup B$ "union" C occurs when at least one of A & oc

$D = A \cap B$ "intersect"

$A \cap B \cap C = (A \cap B) \cap C$

$\bar{A}$, A', or A^c occurs when A doesn't occur (complement of A

$A \leftrightarrow$	1	1	0	0
$B \leftrightarrow$	0	1	0	1
$A \cup B \leftrightarrow$	1	1	0	1
$A \cap B \leftrightarrow$	0	1	0	0

$\overline{(\bar{A})} = A$ | $\overline{(\overline{(\bar{A})})} = \bar{A}$

$A \cancel{+} B \qquad A \cup B$

A^c = complement of $A = \overline{A}$.

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

$(A_1 \cup \dots \cup A_n)^c = A_1^c \cap \dots \cap A_n^c$

$(A_1 \cap \dots \cap A_n)^c = A_1^c \cup \dots \cup A_n^c$

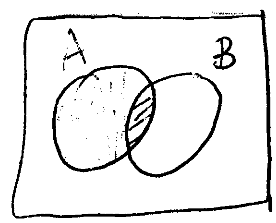
"A implies B" : $A \subset B$ (every time A occurs, B occurs too).

$\quad \| \; A \subset S$.

$\quad \phi \subset A$

DeMorgan Identities

$A = (A \cap B) \cup (A \cap B^c)$



$B = (B \cap A) \cup (B \cap A^c)$

A & B are mutually disjoint if they never occur

A, B are disjoint $\Leftrightarrow A \cap B = \phi$

<u>Probability</u>: (S is the entire sample space)

for every event A from my sample space I associate a # $p(A)$

i) $0 \le p(A) \le 1$

ii) $p(S) = 1$

iii) $A \cap B = \phi \Rightarrow p(A \cup B) = p(A) + p(B)$

$S = S \cup \phi$

$$p(A^c) = 1 - p(A)$$
$$A \cup A^c = S$$
$$p(A) + p(A^c) = 1$$

$$p(A_1 \cup A_2 \cup A_3 \cup \dots \cup_n) = \sum_{i=1}^{n} p(A_i)$$

prove: if A implies B, then $p(A) < p(B)$?
$(p(A) \not> p(B)$

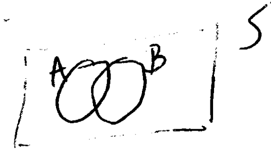
prove this: $A \subset B \implies p(A) \leq p(B)$?

$$\cancel{p(A) + p(B) = 1}$$
$$\cancel{p(A) = 1 - p(B)}$$
$$\cancel{B = A \cup A^c}$$
$$\cancel{p(B) = p(A) + p(A^c)}$$

$$B = (B \cap A) \cup (B \cap A^c) = A \cup (B \cap A^c)$$
so $p(B) = p(A) + p(B \cap A^c) \geq p(A)$. $\blacksquare$

Theorem: For any A, B.
$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

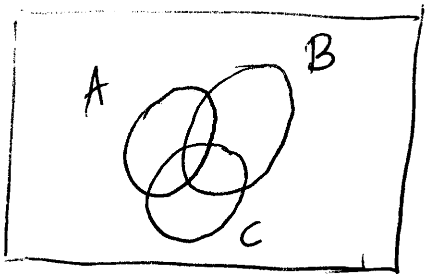
$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$



$$p(A \cup B) = p(A \cap B) + p(A \cap B^c) + p(A^c \cap B) + p(A \cap B) + p(A \cap B) - p(A \cap B) +$$
$$= p(A \cap B) + p(A) + p(A) - p(A \cap B)$$

Δ the probability is the measure of the event ∫ $A \setminus B + B \setminus A$

$P(A \setminus B) + P(B \setminus A)$



$$p(A \cup B \cup C) = p(A) + p(B) + p(C) - p(A \cap B) - p(A \cap C) - p(B \cap C) - p(A \cap B \cap C).$$

Exple. 40% students in swimming class
60% students in soccer class
30% take both
compute the probability of st who take: - at least one class
- exactly one class.

solutᵒ:

$A = \{\text{take swimming class}\}$ $\quad P(A) = \frac{40}{100} = \frac{2}{5}$

$B = \{\text{take soccer class}\}$ $\quad P(B) = \frac{60}{100} = \frac{3}{5}$

$P(A \cap B) = \frac{30}{100} = \frac{3}{10}$

$P(A \cup B) = P(\{\text{take at least one class}\}) = P(A \cup B) = \cancel{\frac{2}{5} + \frac{3}{5}} = 1$

$P(A \cup B) = \frac{2}{5} + \frac{3}{5} - \frac{3}{10} = 1 - \frac{3}{10} = \frac{7}{10}$

↳ $(A \cup B) \setminus (A \cap B)$

$$P((A \cup B) \setminus A \cap B) = P(A \cup B) - P(A \cap B)$$
$$= \frac{7}{10} - \frac{3}{10}$$
$$= \frac{4}{10}$$

$$\boxed{P((A \cup B) \setminus (A \cap B)) = \frac{2}{5}}$$

A die which is "pise" $\Rightarrow$ all possible outcomes have the same probability

S, sample space
finite or infinite
events : $E_1, E_2, E_3, E_4 \ldots$
$S = \bigcup_{i} E_i$

$A \subset S$ (~~A is an event~~ a sub-~~space~~) (A is an event)
$$P(A) = \sum_{i:\, E_i \in A} P(E_i)$$

Ex: $A = \{2, 4, 6\}$
$P(A) = P(2) + P(4) + P(6)$.

a die) $\rightarrow$ $P(E_i) = p \qquad \forall i = 1, 2, \ldots, 6$
$p \in (0, 1) \qquad (p = \frac{1}{6})$

$$P(S) = 1 = \sum_{i=1}^{6} P(E_i) = 6p \quad \rightarrow p = \frac{1}{6}.$$

$\# S = N$
$$1 = \sum_{i=1}^{N} P(E_i) = p \cdot N \qquad p = \frac{1}{N}$$

Uniform probability on a space of N elmts

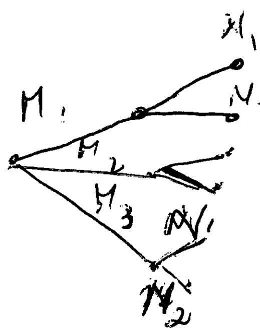
$P(A)$ w/ $A \subset S$
$$P(A) = \sum_{i:\, E_i \in A} P(E_i) = \frac{N(A)}{N}$$ where $N(A)$ is the # of elmts of A.

Ordered Pairs:

$$(O_1, O_2) \neq (O_2, O_1)$$

Prop. If the possible choices for the first elmt ~~are~~ are n_1 ~~& are chose~~ for each choice at the 1st elmt I have n_2 ~~classes~~ choices for the 2nd, ~~the the~~ the # ordered pairs $= n_1 n_2$.

M_1, M_2, M_3 first elmt
N_1, N_2 2nd elmt

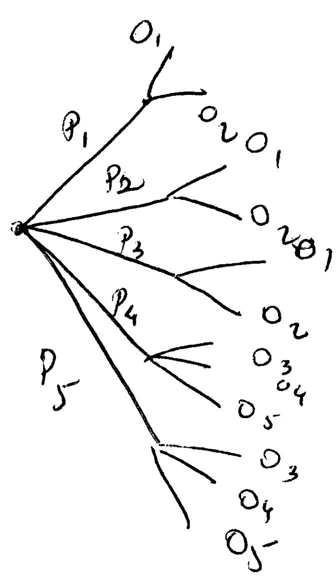


(M_1, N_1)
(M_1, N_2)
(M_2, N_2)

Ex: 2 Hospitals
#1 has 3 pediatricians
2 ostr.
#2 pediatr.
3 ostr.

$\overbrace{P_1, P_2, P_3}^{H_1}, \overbrace{P_4, P_5}^{H_2}$

$\overbrace{O_1\ O_2}^{H_1}\ \overbrace{O_3\ O_4\ O_5}^{H_2}$



Ordered K-TUPLES:

Prop: If I have n_1 choices for the 1st elmt,
for each choice of the 1st elmt, I have n_2 choices for the 2nd

⋮

for each choice of the $(k-1)$ elmt, I have n_k choices for the k^{th} elmt. then # ordered K-Tuples $= n_1 \cdot n_2 \cdot n_3 \cdots n$

ex: die tossed 4 times
$6 \times 6 \times 6 \times 6 = 6^4$.

ex: $(3, 6, T, H)$
$6 \times 6 \times 2 \times 2 = 6^2 \times 2^2$.

$\# S = n$
Choose a subset of K elmts.
$1 \leq K \leq n$.
Permutat^{ns} $\Longleftrightarrow$ Ordered subsets
combinat^{ns} $\Longleftrightarrow$ Unordered subsets
Permutat^{e} has more elmts than combinat^{e}
# of permutat^{e} $= P_{K;n}$

Ex: 7 departments in a college. ; $a_1, a_2, a_3, \ldots, a_7$
$n = 7$
(chair, vice-chair, secretary) e.g.: $(4, 5, 2)$ from 4th, 5th, 2nd dep

$A_1 \cap A_2' \cap A_3'$

$P_{3,7} = 7 \cdot 6 \cdot 5$

so $\boxed{P_{k,n} = n(n-1)\cdots(n-(k-1))}$

$\Updownarrow$

$$\boxed{P_{k,n} = \frac{n!}{(n-k)!}}$$

HW 1: due Monday Ch2: ex $\cancel{2}$, 8, $\cancel{18}$, $\cancel{24}$, $\cancel{26}$, 30, 38, 42

(2.1: 2; 2.2: 18, 24, 26; 2.3: 30, 38, 42)

09/08/09. Set with n elmts

k elmts $k \leq n$

- when order is relevant $\Rightarrow$ permutat[o] (ordered sets)
- combinat[ions] (unordered sets)

$$P_{k,n} = \frac{n!}{(n-k)!}$$

$$C_{k,n} = \#\text{combinat}^{\circ}$$

ex: departmt with 60 faculty members, choose 3 faculty members.

$\{a_1, a_2, a_3\}$

$$C_{3,60} = \frac{P_{3,60}}{3!} = \frac{60!}{(60-3)!\,3!}$$

$$C_{k,n} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

~~A~~ Let S, my sample space

$A \subset S$

A: unconditional event

~~B~~ B has occured

$P(A \mid B)$: conditional probability of A given the fact that B has occured (in general $P(A \mid B) \neq P(A)$)

e.g.: $A = \{\text{student gets grade } A\}$

$P(A)$

$B = \{\text{student got } A \text{ at MATH 141}\}$

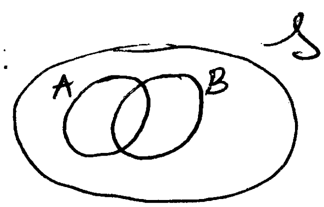
$P(A \mid B)$

in general we can think that $P(A) < P(A \mid B)$

$C = \{\text{student got } D\}$

$P(A \mid C) \leq P(A) \leq P(A \mid B)$.

e.g.:



$P(A \mid B) = c\,P(A \cap B)$

$P(B \mid B) = c\,P(B \cap B) = c\,P(B) = 1 \Rightarrow c = \frac{1}{P(B)}$

Definit: $\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$ w/ $P(B) > 0$

expl: 60% taking swimming classes | 40% soccer | 30% both

$A = \{swimming\}$ $B = \{soccer\}$ $A \cap B$ = both

$P(A) = \frac{3}{5}$ $P(B) = \frac{2}{5}$ $P(A \cap B) = \frac{3}{10}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{10} \times \frac{5}{2} = \frac{3}{4}$$

Multiplicat^e rule:

A, B 2 events with $P(B) > 0$

$$\boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

Expl: 3 ≠ brands of DVD players

Brand 1

Brand 2   they all provide 1 year warranty

Brand 3

25% of Br.1 need repair in 1 year ~~repair~~ time

20% of Br 2

10% for Br 3

1) what's the probability that U buy a DVD of Br 1 & needs repair

Answer: $A_i = \{$ purchased Br. $i\}$

$B = \{$DVD needs repair$\}$

$P(A_1) = \frac{1}{2}$; $P(A_2) = \frac{3}{10}$; $P(A_3) = \frac{1}{5}$

$P(B|A_1) = \frac{1}{4}$; $P(B|A_2) = \frac{1}{5}$; $P(B|A_3) = \frac{1}{10}$

$$P(A \cap B) = P(A|B)\, P(B)$$
$$= P(B|A)\, P(A)$$

$$P(A_2 \cap B) = P(B|A_1) \cdot P(A_1) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

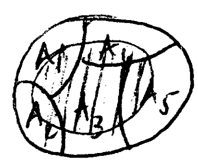
BAYE'S theorem:

S: samplespace

$A_1, A_2, \dots A_n$ events in S

- they are mutually exclusive if they are all disjoint
  $A_i \cap A_j = \emptyset \quad \forall\, i, j = 1 \dots n \quad i \neq j$
- they are exaustive if $\bigcup_{i=1}^{n} A_i = S \quad i \neq j$

e.g.:



$\{B\}$

$A_1 \cup A_2 \cup \quad \cup A_5 = S$

$\{A_1 \dots A_n\}$ is called partit° of S.

Law of Total Propability:

$A_1, \dots, A_n$ mutually exclusive & exaustive w/ $P(A_i) > 0 \ \forall i$

B event

$$P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

$$B = \bigcup_{i=1}^{n} B \cap A_i$$

$$\underset{i \neq j}{A_i \cap A_j} = \emptyset \Rightarrow (B \cap A_i) \underset{i \neq j}{\cap} (B \cap A_j) = \emptyset$$

$$B = \bigcup_{i=1}^{n} B \cap A_i$$

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$$

Exple continued:

2) Compute the probability that U buy a DVD player which needs repair.

Answer. $P(B) = P(B|A_1)\,P(A_1) + P(B|A_2)\,P(A_2) + P(B|A_3)$

$= \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{3}{10} + \frac{1}{10}\cdot\frac{1}{5} =$

3) If I am returning a DVD player, what's the ~~prob~~ probab. that it w of B2?

$$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{P(B|A_2)\ P(A_2)}{P(B)}$$

Baye's theorem:

⚠ $A_1 \dots A_n$: partit

B event w/ $P(B) > 0$

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)\ P(A_j)}{\sum_{i=1}^{n} P(B|A_i)\,P(A_i)}$$

j is fixed

Independence:

Events A, B

$$P(A|B) = \frac{P(A \cap B)}{P(B}$$

provided $P(A) > 0$ & $P(B) > 0$

When $P(A|B) = P(A)$, we say that A and B are independent then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)} = \frac{P(A) \cdot P(B)}{P(A)} = P(B)$$

Fact: If A & B are independent, then also A and B' are independent, & also A' & B'.

A, B independent $\Rightarrow$ A, B' are

$$P(B'|A) = P(B')$$

$$P(B'|A) + P(B|A) = \frac{P(B' \cap A)}{P(A)} + \frac{P(B \cap A)}{P(A)}$$

$$= \frac{P(B' \cap A) + P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$P(B'|A) = 1 - P(B) = P(B')$$

ex: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$ $\quad$ $P(1) = P(6) = .10$

$B = \{1, 2, 3\}$ $\quad$ $P(2) = P(5) = .15$

$C = \{3, 4, 5\}$ $\quad$ $P(3) = P(4) = .25$

$P(A) = .15 + .25 + .10 = \frac{1}{2}$

$P(B) = .10 + .15 + .25 =$

$P(C) = .15 + .25 + .25$
$= .80$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(2) = 2$$

$0.30 \neq \frac{1}{2}$

⚠ if $A \cap B = \phi$ $\quad P(A|B) = \frac{P(A \cap B)}{P(B)} = 0 \neq P(A)$

so here A & B are disjoint but A & B are **not** indepen

A & B not independent:

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P\{2,4\}}{8} \cdot 10 = \left(\frac{15}{100} + \frac{25}{100}\right)\left(\frac{10}{8}\right) = \frac{40}{80} = \frac{1}{2}$$

<u>Proposit</u>: A & B are independent $\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

in particular, if U have A & B and $P(A) = 0$, $P(B) \neq 0$,

then A & B are independent

cuz $P(A \cap B) = P(A|B) \cdot P(B)$ with $P(A|B) = P(A)$

<u>Def</u>: Let events $A_1, A_2, A_3, \dots, A_n$

They are independent if $\forall\, 2 \leq k \leq n$ $\forall\, i_1 \cdots i_k \in \{1, 2, \dots, n\}$

$i_l \neq i_j \quad l \neq j$

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

<u>ex</u>: $S = \{1, 2, 3, 4\}$

$P_i = \frac{1}{4}$

$A = \{1, 4\}$ $\quad B \cap C = \{4\}$ $\quad P(A) = P(B) = P(C) = \frac{1}{2}$

$B = \{2, 4\}$ $\quad A \cap B = \{4\}$ $\quad P(A \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(C)$

$C = \{3, 4\}$ $\quad A \cap C = \{4\}$ $\quad P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B)$

$A \cap B \cap C = \{4\}$ $\quad P(A \cap B \cap C) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C)$

so these events are not independent as a family,

they are only independent two by two.

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! Enveloppe for hmwk

Exple: 2 pumps working in parallel independently
The probability that only the first one fails is $\frac{1}{10}$
— " — " — " — " — 2nd one fails is $\frac{5}{100}$
what is the prob. that the system fails?

solut: $A_1 = \{\text{1rst fails}\}$

$A_2 = \{\text{2nd fails}\}$

$P(A_1 \cap A_2) = ?$

$P(A_1 \cap A_2') = \frac{1}{10}$

$P(A_1' \cap A_2) = \frac{5}{100}$

$$\overbrace{P(A_1 \cap A_2)}^{x} = P(A_1) \cdot P(A_2)$$

$$= \left[P(A_1 \cap A_2) + P(A_1 \cap A_2')\right] \cdot \left[P(A_2 \cap A_1) + P(A_2 \cap A_1')\right]$$

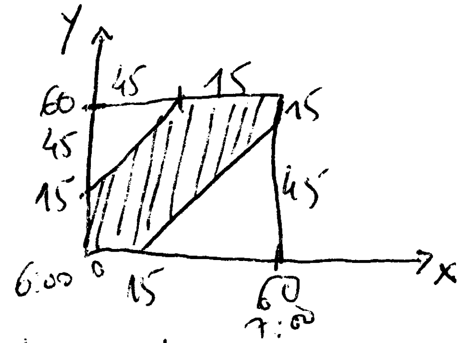
$$= \left(x + \frac{1}{10}\right)\left(x + \frac{5}{100}\right)$$

So $x = \left(x + \frac{1}{10}\right)\left(x + \frac{5}{100}\right)$

⟹ Exple: A & B have an appointment [@] 6:00pm – 7:00pm & agree that the one comes earlier should wait the other one for 15 mins & then leave. What is the proba. that A & B meet?

outcome is 15 mn)

Solut:



$|x - y| \leq 15$

$$P = \frac{\text{Area of strip}}{\text{Area of square}} = \frac{60^2 - 45^2}{60^2} = \frac{7}{16}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

9:30
10:00
10:20

$$3 \left\{ \begin{matrix} H\,T\,T \\ T\,H\,T \\ T\,T\,H \end{matrix} \right.$$

Exple: Toss a coin 3 times. Find P for
(1) $A_1 = \{H \text{ once}\}$

(2) $A_2 = \{H \text{ at least once}\}$

solut: (1) $P(A_1) = \cancel{\frac{1}{3!}} = \cancel{\frac{1}{6}} = \frac{3}{8}$

(2) $P(A_2) = \cancel{1 - \frac{1}{3!}} = \cancel{\frac{5}{6}} = \frac{7}{8}$

$A_2 = \{(H,T,T), (THT), (TTH), (HHT), (HTH), (THH), (HHH)\}$

$S = \{(8 \text{ outcomes})\}$

Exple: There are N products, M defectives
pick n out of N. $P\{\text{get exactly } k \text{ defectives}\}$

$\binom{N-M}{n-k}$, $\binom{M}{k}$

$\binom{N}{n}$

solut. pick n from N: $\binom{N}{n}$ combinat^os
pick k from M: $\binom{M}{k}$ combinat^os

pick $n-k$ from $N-M$: $\binom{N-M}{n-k}$ combinat^os

So $$\boxed{P = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}}$$

$N \times (N-1)(N-2)$

$\binom{N}{1}\binom{N-1}{1}\binom{N-2}{1}$

$N-n-1 \quad N!$

$(N-n+1) \quad N^n \quad \frac{N!}{n}$

Ex: A white & black ball are in a box.
pick a ball from the box, if it is white, put another white with this white ball into the box. If black, stop. what is the probability that the black is not picked after n times.

solut: $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdots \frac{n-1}{n} \cdot \frac{n}{n+1} = \boxed{\frac{1}{n+1}}$.

$A_i = \{\text{pick W at } i^{th} \text{ time}\}$

$A = \{\text{No B after n times}\}$

$$P(A) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2) \cdots$$

$$= P(A_n|A_1 \cdots A_{n-1})\, P(A_{n-1}|A_1 \cdots A_{n-2}) \cdots P(A_2|A_1) P(A_1)$$

$$= \frac{n}{n+1} \times \frac{n-1}{n} \cdots \frac{1}{2} = \frac{1}{n+1}$$

$$P(A) = \frac{1}{n+1}$$

Ex: put n balls randomly into N boxes $(N \geq n)$. find the prob that there are at most one ball in each box.

solut: There are $\underbrace{N \times N \times \cdots \times N}_{n}$ ways to input the balls (sample space)

pick n boxes from N boxes: $\binom{N}{n}$

then put n balls in the n boxes: $n!$

& $\boxed{P = \frac{\binom{N}{n} n!}{N^n}}$

TA ~~sess~~ Exo1:

State	1	2	3
F	3	7	5
M	7	8	20
Total	10	15	25

Randomly pick a state & then randomly choose 2 applicants

(1) Prob. that 1^{st} applicant is a Female

(2) Given 2^{nd} is a Male, what is prob. that the 1^{st} is a Female?

$A_i = \{$ the ith Applicant is Female $\}$

$B_i = \{$ the applicant is from the ith state $\}$

$$P(A_i) = P(A_1|B_1)\,P(B_1) + P(A_1|B_2)\,P(B_2) + P(A_1|B_3)\,P(B_3)$$

$$= \frac{3}{10}\times\frac{1}{3} + \frac{7}{15}\times\frac{1}{3} + \frac{5}{25}\times\frac{1}{3} = \frac{29}{90}$$

(2) $P(A_1|A_2^c) = \dfrac{P(A_1 \cap A_2^c)}{P(A_2^c)}$

$P(A_2^c) = \frac{1}{3}\left(\frac{7}{10} + \frac{8}{15} + \frac{20}{25}\right) = \frac{6}{9}$

$P(A_1 \cap A_2^c) = \sum_i P(A_1 A_2^c | B_i)\, P(B_i)$

$= \frac{1}{3}\left(\frac{3}{10}\times\frac{7}{9} + \frac{7}{15}\times\frac{8}{14} + \frac{5}{25}\right.$

$= \frac{2}{9}$

$P(A_1/A_2^c) = \dfrac{2/9}{61/90}$

Exo2: Prob that a person has swine ~~flu~~ flu is $1.5\% = 0.015$

$C = \{$ a person has S.P. $\}$

$A = \{$ result of the Assay is pos

$P(A|C) = 0.90$

$P(\bar{A}|\bar{C}) = 0.95$

$P(A|\bar{C}) = 0.05$

what is $P(C|A)$?

$P(C|A) = \dfrac{P(A \cap C)}{P(A)}$

but $P(A \cap C) = P(A|C) \cdot P(C)$

$P(A \cap C) = 0.9 \times 0.015$

$$P(A) = P(A \cap C) + P(A \cap \bar{C})$$
$$= P(A|C)\,P(C) + P(A|\bar{C})\,P(\bar{C})$$
$$P(A) = 0.9 \times 0.015 + 0.05 \times (1 - 0.015)$$

09/17/09 $\mathcal{S}$, P $X : \mathcal{S} \longrightarrow D \in \mathbb{R}$

pmf $p(x) = P(X = x)$ $x \in \mathbb{R}$

$F(x) = P(X \leq x)$ (cumulative Distributional Fc

$$F(x) = \sum_{\substack{y \in D \\ y \leq x}} p(y)$$

$$\{X \leq x\} = \bigcup_{\substack{y \leq x \\ y \in D}} \{X = y\} = \sum_{\substack{y \leq x \\ y \in D}} P(\{X = y\})$$

$$= \sum_{\substack{y \leq x \\ y \in D}} p(y)$$

Ex:

x	1	2	3	4
$p(x)$	.4	.3	.2	.1

- $x < 1$

  $F(x) = P(X \leq x) = 0$

- $x = 1$

  $F(1) = P(X \leq 1) = P(1) = .4$

- $x \in [1, 2)$

  $F(x) = P(X \leq x) = P(1) = .4$

- $x \in [2, 3)$

  $F(x) = P(X \leq x) = P(\{x = 1\} \overset{(or)}{\cup} \{x = 2\})$

  $= p(1) + p(2)$

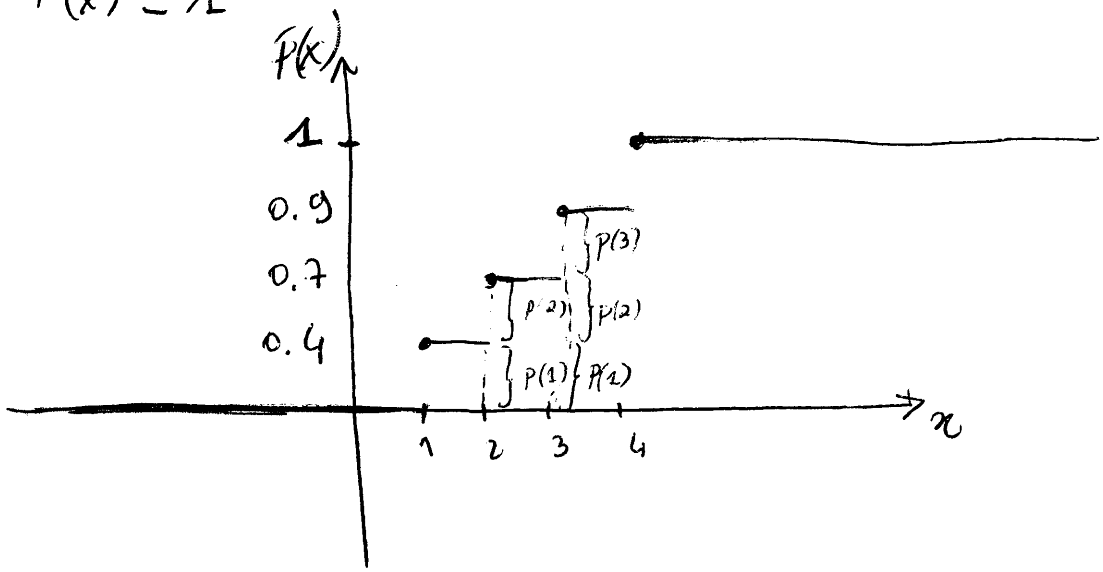
- $x \in [3,4)$

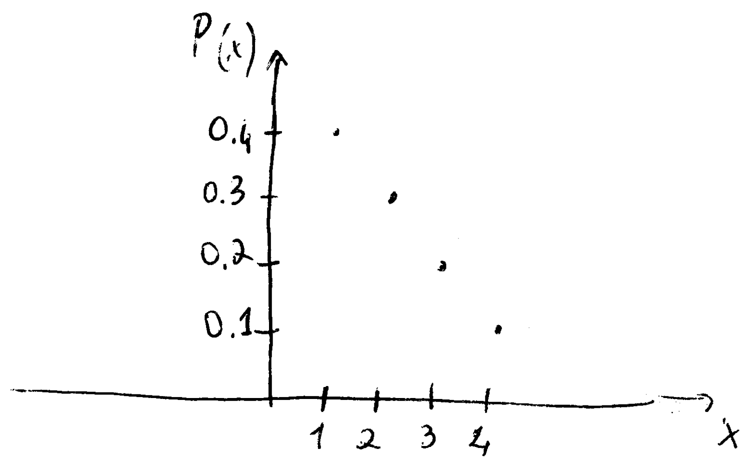
$$F(x) = p(1) + p(2) + p(3)$$

- $x \geq 4$

$$F(x) = 1$$

<u>Picture</u>



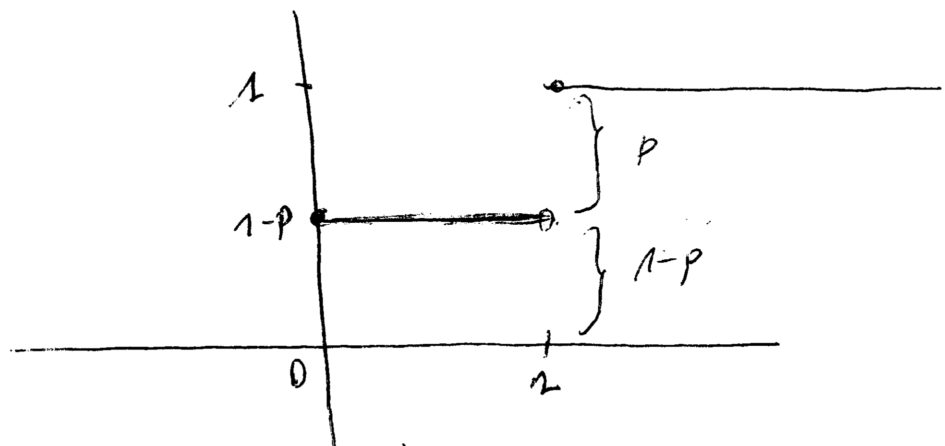


8, X Bernoulli of parameter p

pmf $P(X=1) = p$ $P(X=0) = 1-p$

$F(x) = 0 \quad x<0$

$F(x) = 1-p \quad x \in [0,1)$ $F(x) = 1 \quad x \geq 1$



$$p(x) = \begin{cases} p(1-p) & x \leq 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

$p \in (0,1)$

$F(x) = 0$

$F(x) = \sum_{y \leq x} p(y)$

$F(x) = \sum_{y \leq x} p(y) = \sum_{y=1}^{x} p(1-p)^{y-1}$

$= p \sum_{y=1}^{x} (1-p)^{y-1} = p \sum_{y=0}^{x-1} (1-p)^{y} = p \left[\frac{1-(1-p)^{x}}{1-(1-p)} \right]$

$$\boxed{a \neq 1 \qquad \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}}$$

$a = 1-p$

$k = y$

$n = x+1$

$$= \underline{p\left(1-(1-p)^x\right]}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1-(1-p)^k & x \in [k, k+1) \end{cases}$$

Exple:

$$F(x) = \begin{cases} 0 & x < 1 \\ .4 & x \in [1,2) \\ .7 & x \in [2,3) \\ .9 & x \in [3,4) \\ 1 & x \geq 4 \end{cases}$$

x	1	2	3	4
$p(x)$	.4	.3	.2	.1

$p(2) = p(X=2) = P(X \leq 2) - p(X \leq 1) = .7 - .4 = .3 = F(2) - F($

$p(3) = p(X=3) = F(3) - F(2) = .2$

⚠ Proposit°: Let $a \leq b$ 2 #s

then $P(a \leq X \leq b) = F(b) - F(a^-)$

where a^- is the largest value of X smaller than a.

In particular, if $X: S \to \mathbb{N}$ & $a, b \in \mathbb{N}$

$$\mathbb{P}(a \leq X \leq b) = P(\{X = a\} \cup \{X = a+1\} \cup \cdots \cup \{X = b\})$$

$$= p(a) + p(a+1) + \cdots + p(b)$$

$$= [F(a) - F(a-1)] + [F(a+1) - F(a)] + \cdots + [F(b) - F(b$$

$$P(a \leq X \leq b) = F(b) - F(a-1)$$

⚠ $P(a < X \leq b) = F(b) - F(a)$

Expected Mean of a r.v.:

Def: $EX = \sum_{x \in D} x\, p(x)$

expected value / mean of X

X	x_1	x_2	...	x_n
$p(x)$	$\frac{1}{n}$	$\frac{1}{n}$		$\frac{1}{n}$

$$EX = \frac{1}{n} \sum_{i=1}^{n} x_i$$

X	x_1	x_2	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	...	$p(x_n)$

$$EX = \sum_{i=1}^{n} x_i\, p(x_i)$$

ex: X Bernoulli variable of parameter p

$p(1) = p \quad p(0) = 1-p$

$EX = 0(1-p) + 1 \cdot p$

$\boxed{EX = p}$

ex:

$$p(x) = \begin{cases} p(1-p)^{x-1} & \text{for } x = 1,2,\dots \\ 0 & \text{otherwise} \end{cases}$$

$$EX = \sum_{x=1}^{+\infty} p(1-p)^{x-1} \cdot x$$

$$= p \sum_{x=1}^{+\infty} x(1-p)^{x-1}$$

we have $\frac{d}{dp}\left[(1-p)^x\right] = -x(1-p)^{x-1}$

so

$$EX = -p \sum_{x=1}^{+\infty} \frac{d}{dp}\left[(1-p)^x\right]$$

$$= -p \frac{d}{dp} \sum_{x=1}^{+\infty} (1-p)^x$$

$$= -p \frac{d}{dp} \sum_{x=0}^{+\infty} (1-p)^x$$

$$= -p \frac{d}{dp}\left[\frac{1}{1-(1-p)}\right]$$

$$= -p \frac{d}{dp}\left(\frac{1}{p}\right)$$

$$= -p\left(\frac{-1}{p^2}\right)$$

$$\boxed{EX = \left(\frac{1}{p}\right)}$$

Remark: If $X : \mathcal{S} \longrightarrow \{x_1, x_2, \dots, x_n\}$
then $EX < +\infty$

$$EX = \sum_{i=1}^{+\infty} x_i \, p(x_i)$$

example

$$p(x) = \begin{cases} \frac{K}{x^2} & x = 1, 2, 3, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$\text{w/ } K \sum_{x=1}^{+\infty} \frac{1}{x^2} = 1 \rightarrow K = \frac{1}{\sum_{x=1}^{+\infty} \frac{1}{x^2}} = \frac{6}{\pi^2}$$

$$EX = \sum_{x=1}^{+\infty} x \, p(x) = K \sum_{x=1}^{+\infty} \frac{1}{x} = +\infty$$

example:

x	2	4	6
$g(x)$	.5	.2	.3

$h : \mathbb{R} \to \mathbb{R}$

$h(x) = 2x + x^2 + 1$

$h(X) : \mathcal{S} \to D^* = h(D)$

y	9	25	49
$p(y)$	.5	.2	.3

$$Eh(X) = 9(.5) + (25)\cdot(.2) + (49)(.3)$$

Let

$h(x) = ax + b \quad a, b \in \mathbb{R}$

$$
\begin{aligned}
E(h(x)) &= E(ax+b) \\
&= \sum_{x \in D} (ax+b)\, p(x) \\
&= \sum_{x \in D} ax\,p(x) + \sum_{x \in D} b\, p(x) \\
&= a \sum_{x \in D} x\, p(x) + b \sum_{x \in D} p(x)
\end{aligned}
$$

$$E(h(x)) = E(ax+b) = a\,EX + b$$

If $b = 0$, $E(ax) = a\,EX$

If $a = 1$, $E(x+b) = EX + b$

<u>exple</u>: I)

x	2	3	4
$p(x)$	0.3	0.4	0.3

X_I

$$EX_I = 2(0.3) + 3(0.4) + 4(0.3) = 0.6 + 1.2 + 1.2 = 3$$

II)

x	1	2	6
$p(x)$	0.6	0.3	0.3

X_{II}

$$EX_{II} = .6 + 2(.3) + 6(.3) = .6 + .6 + 1.8 = 3$$

$X : \mathcal{S} \to D \qquad EX < +\infty$

$$E\left[(X - EX)^2\right] = V(X) = \sigma_X^2$$

Variance of X

Compute variances

x	1	-1
$p(x)$	$1/2$	$1/2$

$EX = 0$

x	10^6	-10^6
$p(x)$	$1/2$	$1/2$

09/24/09.

$\mathcal{S} : \mathbb{P}$

$X : \mathcal{S} \to D \qquad p(x) \qquad$ pmf

$$EX = \sum_{x \in D} x\, p(x)$$

$$V(X) = E\left[(X - EX)^2\right]$$

$$\overset{\parallel}{\sigma_X^2}$$

variance of X

Property: $V(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu_X^2$

proof: $V(X) = E\left[(X-\mu)^2\right] = \sum_{x \in D} (x-\mu)^2 p(x)$

$$= \sum_{x \in D} \left(x^2 + \mu^2 - 2\mu x\right) p(x)$$

$$= \sum_{x \in D} x^2 p(x) + \mu^2 \sum_{x \in D} p(x) - 2\mu \sum_{x \in D} x\, p(x)$$

⚠ $E(aX+b) = aEX + b.$

X = random variable
x = values taken by the random variable

$$= E(x^2) + \mu^2 - 2\mu^2 = E(x^2) - \mu^2$$

$$E(x^2) - \mu^2 = V(x) \geq 0 \; ; \qquad \boxed{E(x^2) \geq (EX)^2}$$

⚠ $\boxed{\begin{array}{l} Eh(x) = \sum h(x)\, p(x) \\ h(x) = (x-\mu)^2. \end{array}}$

$$X : S \longrightarrow D \qquad h : D \longrightarrow \mathbb{R}$$

$$Eh(x) = \sum_{x \in D} h(x)\, p(x)$$

$$V(h(x)) = E\left[\left(h(x) - E(h(X))\right)^2\right]$$

$$= \sum_{x \in D} \left(h(x) - Eh(X)\right)^2 p(x)$$

e.g.: $h(x) = ax + b$

$$V(aX+b) = \sum_{x \in D} (ax + b - aEX - b)^2 p(x)$$

$$= \sum_{x \in D} a^2 \left(x^2 + (EX)^2 - 2xEX\right) p(x)$$

$$= a^2 \sum_{x \in D} x^2 p(x) + a^2 (EX)^2 \sum_{x \in D} p(x) - 2a^2 EX \sum_{x \in D} x\, p(x)$$

$$= a^2 E(X^2) + a^2 (EX)^2 - 2a^2 (EX^2)$$
$$= a^2 \left(E(X^2) - (EX)^2\right) = a^2 V(X).$$

Bernoulli Distribut^{k}: 2 outcomes : . success with probability p
. Failure with probability $1-p$.

$$\boxed{V(aX+b) = a^2 V(X)}$$

$$E(aX) = aEX.$$

35
49

Bernoulli.

$EX = p$

$V(X) = E(X-p)^2 = \sum_{x=0,1} (x-p)^2 p(x) = p^2(1-p) + (1-p)^2 p = p(1-$

so $V(X) = p(1-p)$.

Binomial Experiment: (simple exple is when U toss a coin $\triangle$)

① It consists of n smaller experiments called trials.

② the result of each trial is independent of the result of all oth
trials

③ All trials have only 2 possible results : S or F (Success or Fail

④ All trials have the same probability of success denoted $p \in$

Exple1: 50 restaurants ; 15 are not good
35 are good

5 inspectors . 5 trials

the result of the ith trial is S if the ith restaurant is

$S_{ith} = \{$ith restaurant is good$\}$

$P(S_1) = \frac{35}{50} = \frac{7}{10}$

$P(S_i) = \frac{35}{50}$.

$P(S_2) = P(S_2 \cap S_1) + P(S_2 \cap F_1)$
$= P(S_2|S_1)P(S_1) + P(S_2|F_1)P(F_1)$
$= \frac{34}{49}\left(\frac{35}{50}\right) + \left(\frac{35}{49}\right)\left(\frac{15}{50}\right) = \boxed{\frac{35}{50}} = P($

②

$$P(S_5 \mid S_1 S_2 S_3 S_4) = \frac{31}{46}$$

$$P(S_5 \mid F_1 F_2 F_3 F_4) = \frac{35}{46}$$

there is not independence ; the result of each trial
⟹ Not a binomial exper

Exple 2: 500,000 people w/ a driving licence
400,000 of them are insured
choosing 10 of people w/ a driving licence

- No independence) cuz U extract without replacemt (the names) but U can approximate an experiment without replacemt to an experiment with replacemt (binomial) when the size is really really big (of the populat) (like here).

Binomial experiment with n trials $<^{S}_{F}$

$P(S) = p$

Binomial Random Variable

$X = \#$ of successes in n trials

$X : \mathcal{S} \longrightarrow \{0, 1, 2, \ldots, n\}$

Exple:

$n = 3$	x	Probability (cuz independent)
SSS	3	p^3
SSF	2	$p^2(1-p)$
SFS	2	$p^2(1-p)$
FSS	2	$p^2(1-p)$
SFF	1	$p(1-p)^2$
FSF	1	$p(1-p)^2$
FFS	1	$p(1-p)^2$
FFF	0	$(1-p)^3$

$p(3) = p^3$

$p(2) = 3p^2(1-p)$

$p(1) = 3p(1-p)^2$

$p(0) = (1-p)^3$

⚠ $P(E \cup C \cup L) = P(E) + P(C) + P(L) - P(E \cap C) - P(E \cap L) - P(C \cap L) + P(E \cap C \cap L$

$P(E \cup C \cup L) = 1 - P(\bar{E} \cap \bar{C} \cap \bar{L})$

$F(-\infty) = 0$
$F(+\infty) = 1$

exo 2.106:

order	G	B	Judge
123	60%	.10	G
132	.25	.20	B
312	.15	.70	B
P	0.25	.075	

a) $P(G \mid 123) = \dfrac{P(G \cap 123)}{P(123)} = \dfrac{P(123 \mid G)P(G)}{P(123 \mid G)P(G) + P(123 \mid B)P(B)}$

$= \quad = \dfrac{0.6 \times 0.25}{0.6 \times 0.25 + 0.10 \times 0.75}$

$= 67\%$

$P(B \mid 123) = 1 - P(G \mid 123) = 33\%$

Judge G cuz there's more chances it's G, (67%) (1rst c

b) $P(G \mid 132) = \dfrac{P(G \cap 132)}{P(132)} = \dfrac{P(132 \mid G)P(G)}{P(132 \mid G)P(G) + P(132 \mid B)P(B}$

$= 29.4\%$

c) $P(\text{error}) = P(B \text{ as } G) + P(G \text{ as } B)$

$= P(123 \mid B)P(B) + P(132 \mid G)P(G) + P(312 \mid G)P$

$= \ldots$

$p > \frac{4}{17}$

d) $P(G \mid 123) > P(B \mid 123)$

$P(G \mid 132) > P(B \mid 132)$

$P(G \mid 312) > P(B \mid 312)$

$P(G \mid 123) = \dfrac{P(123 \mid G)P(G)}{P(123 \mid G)P(G) + P(123 \mid B)P(B)} = \dfrac{0.6p}{0.6p + 0.1(1-p)} > 0.5$

Right continuousness of F (cdf):

$$F(\cancel{x}) = F(x^+)$$

$$P(X \leq x) = P(X \leq x^+)$$

but $P(X \geq x) \neq P(X \geq x^+)$

$\Downarrow$ for $X > x^+$ but not for $X = x^+$

so <u>not</u> left continuous

toss a coin n times. the proba that U get k heads in the n times is :

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Exo: 10 defectives in 100 products. 5 products are chosen with puting back.

what's the proba?

(1) $P(\text{choose } 1 \text{ defective}) = \binom{5}{1}(0.1)^1 (0.9)^4$

(2) $P(\text{choose less than } 3 \text{ defectives}) = P(X=1) + P(X=2) + P($

$$= \binom{5}{1}(0.1)^1 (0.9)^4 + \binom{5}{2}(0.1)^2(0.9)^3 + \binom{5}{0}(0.1)^0(0.9)^5$$

~~B = Bernoulli~~ $B(n,p)$: ~~Bernoulli~~ Binomial w/ parameters n

~~Bernoulli~~

~~Toss~~ Success or Fail

$n = 3$. $X = \#$ of S's in n trials

$p(0) = (1-p)^3$

$p(1)\ \{(1-p)^2\, p\}(3)$

$p(2) = 3\,p^2(1-p)$

$p(3) = p^3$

Let n : $X = 0, 1, 2, 3, \ldots, n$

$k = 0, 1, 2, \ldots, n$

$p(k) = P(X = k)$

$n \begin{cases} k & S \\ n-k & \end{cases}$

$$p(a_1, a_2, \ldots, a_n) = p^k (1-p)^{n-k}$$

e.g. $P(SSF) = P(S)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$X \sim B(n,p)$ $n \in \mathbb{N}$ (Binomial distribution)

& $P(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{for } k = 0, 1, 2, 3, \ldots, n \\ 0 & \text{otherwise} \end{cases}$

For $n = 1$, Binomial = Bernoulli (cuz just 1 trial)

Proposit: if $X \sim B(n,p)$ (if X is a binomial distribut w/ parameters n & p)
then $EX = np$ & $V(X) = np(1-p)$
expected value — variance

$$EX = \sum_{x=1}^{n} \cancel{x} \binom{n}{x} p^{\cancel{x}} (1-p)^{n-x}$$

$$\frac{n!}{x!(n-x)!} = \binom{n-1}{x-1}$$ so $\binom{n}{x} = \binom{n-1}{x-1}$

$$V(X) = E(X^2) - (EX)^2 = EX^2 - n^2p^2$$ for Binomial dis

HyperGeometric Distribute:

① A pop. of N individuals
② each individual can be labeled either with S or F there are M S's.
③ A sample of n individuals is randomly taken & any choice of n individuals has the same probability.

X = # of S's in the chosen sample of n individuals.
$h(x, n, N, M) = P(X = x)$

($M \geq n$ max $X = n$, $M < n$ max $X = M$)

if $N - M \geq n \Rightarrow X \geq 0$; if $N - M < n \Rightarrow X \geq n - (N - M)$

e.g.

16 red balls (S)

12 black balls (F)

$N = 28 \qquad M = 16$

I extract 8 balls $\quad (n = 6)$

$n = 8$

P(~~extract~~ 8 red balls) = ~~1~~

⚠ $P(\text{extract } 2 \text{ red balls}) = \dfrac{\binom{16}{2}\binom{12}{6}}{\binom{28}{8}}$

⚠ $\boxed{h(2;8,28,16)}$

⚠

N 2 groups S F

indiv of groups S is M

select a couple of n indiv.

X = # indiv of group S

X is an hypergeometric rv

$$h(x;N,M,n) = P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

<u>prop.</u>: $EX = n\frac{M}{N}$ $\quad Var(X) = \frac{(N-n)}{N-1}$

⚠ the # of trials is fixed in a binomial

<u>Negative Binomial</u>:

The # of successes is fixed (not the # of trials)

1. Experiment consists in a sequence of indep. trials
2. the result of each trial is either F or S
3. At each trial the proba of S is p.
4. the experiment stops when you get r S's

X = # failures before the rth success

$x = \# F$
$r = \# S$

$X = 5 \quad nb(x; p, r) = P(X = x)$

$r = 3 \quad x = 0, 1, 2 \ldots$

$$P(X = x) = P\left(\begin{array}{l} r-1 \text{ S's in } x+r-1 \text{ trials and S from the } x+r\text{th trial} \end{array}\right)$$

$$= \binom{x+r-1}{r-1} p^{r-1}(1-p)^x p$$

→

$$= \binom{x+r-1}{r-1} p^r (1-p)^x$$

Poisson Distr.:

Def: X has Poisson Distr w/ parameter $\lambda > 0$ if

$$P(X = x) = \begin{cases} e^{-\lambda}\left(\frac{\lambda^x}{x!}\right) & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x=0}^{+\infty} e^{-\lambda} \frac{\lambda^x}{x!} = 1.$$

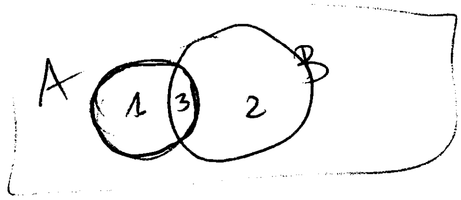
prop.: take $n \in \mathbb{N}$ $\quad p_n \in (0,1)$

s.t. $n p_n \to \lambda > 0$, as $n \to \infty$

then $b(x, n, p_n) \to p(x, \lambda)$ $\quad x = 0, 1, 2 \ldots$

$$b(x, p, n) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$=$$



$P(A) = 1 + 3$
$P(B) = 2 + 3$
$1 = 2$

3 Axioms for Probability:

1)- Positivity $P > 0$
2)- Additivity
3)- Normality : $P(\emptyset) = 0$
$P(\Omega) = 1$

Sample space Ω
Event $E \subset \Omega$
$P : E \subset \Omega \to p \in [0, 1]$

Ex 1: $P(\bar{A} \cap \bar{B}) = \frac{1}{9}$

$P(A \setminus B) = P(B \setminus A)$

$P(A \cap \bar{B}) = P(B \cap \bar{A})$

$P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$

$P(A \cap \bar{B}) =$

$P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B})$

$P(B \cup \bar{A}) = P(B) + P(\bar{A}) - P(B \cap \bar{A})$

$P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(B \cap \bar{A})$

$P(A) + P(\bar{B}) - P(A \cup \bar{B}) = P(B) + P(\bar{A}) - P(B \cup \bar{A})$

$P(A) = P(B) + P(\bar{A}) - P(B \cup \bar{A}) + P(A \cup \bar{B}) - P(\bar{B})$

$\cancel{= P(B) - P(\bar{B}) + P(\bar{A}) + 1 - P(A)}$

$2P(A) = 2P(B) - 1 + P(\bar{B}) - P(B \cup \bar{A}) + P(A \cup \bar{B})$

$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B}) =$

$P(A \cap B) = P(A) \cdot P(B)$ becuz
A & B are indep

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$1 - P(\bar{A} \cap \bar{B}) = x + x - x^2 = 8/9$

$x^2 - 2x + \frac{8}{9} = 0$

$x = 2/3$
$x = 4/3$
cuz $P > 0$

~~$P(A) = P(A|B_i) P($~~

$P(A) = \sum P(A|B_i) P(B_i)$

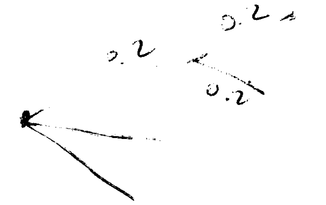
$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j) P(B_j)}{\sum P(A|B_i) P(B_i)}$

when events are independent

$P(A \cap B) = P(A) \cdot P(B)$

$P(A|B) = P(A)$.

Hit



Ex 2: $P(A) = 0.4$

$P(B) = 0.5$

$P(C) = 0.7$

$P(\text{Hit Down}) = 0.2$ if hit by ~~2~~ people

$P(\text{Hit Down}) =$

$$P(H) = P(H|A) + P(H|B) + P(H|C) + P(H|A\cap B) + P(H|A\cap C) + P(H|B\cap C) + P(H|A\cap B\cap C)$$

$$P(H) = \frac{P(H\cap A)}{P(A)} + \frac{P(H\cap B)}{P(B)} + \frac{P(H\cap C)}{P(C)} + \frac{P(H\cap A\cap B)}{P(A\cap B)} + \frac{P(H\cap A\cap C)}{P(A\cap C)} + \frac{P(H\cap B\cap C)}{P(B\cap C)} + \frac{P(H\cap A\cap B\cap C)}{P(A\cap B\cap C)}$$

$$P(H) = \frac{(0.4)(0.2)}{0.4} + \frac{(0.2)(0.5)}{(0.5)} + \frac{(0.2)(0.7)}{(0.7)} + \frac{(0.4)(0.5)(0.2)}{} +$$

Hit by 1:

$P_1 =$ A hit: $0.4 \times (1-0.5) \times (1-0.7) + (1-0.4)(0.5)(1-0.7) + (1-0.4)(1-0.5)(0.7)$

Hit by 2:

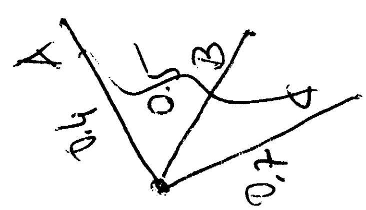
$P_2 =$

$$\boxed{P = 0.2P_1 + 0.6P_2 + (1)P_3}$$

$P_1 = P(A)\cdot(P(B))\cdot(P(C))$

~~P(H|A~~

$$P(A|H) = \frac{P(A\cap H)}{P(H)} =$$



A hit: $P = (0.4)(1-0.5)(1-0.7)$

$P(A) = P(A|B_i)P(B_i)$

0.2 $P(H) =$

2: $P(H_1) =$ rolling down by 1 $P(H_1$

$p(H) =$ go down $P(A_1) =$ hit by $1 =$ |

$2+4+8+16+32+64$... geo sum w/ $r = 2$ $\frac{1-r^{n+1}}{1-r}$

Ex 3: $P(S) = p$

~~$P(T) = (1-p)^m \; p^{n-m}$~~

$$\boxed{P(T) = \binom{n+m-1}{m} p^{n-1} (1-p)^m}$$

Doing a series of experiments independently
success rate is p
what is the probability U exactly
lose m times before n success

Ex 4:

~~$1 = p + 2p + 3p + \dots$~~

$1 = \frac{R(R+1)}{2} p \Rightarrow \boxed{p = \frac{2}{R(R+1)}}$

P_k has

P_k & p

$$\boxed{p = \sum_{k=1}^{\infty} \frac{P_k}{k-1}}$$

A random family has R kids w/ proba
P_k, $R = 0, 1, 2, 3, \dots$
Assume the proba. to be a boy or girl
is equal & independent for each kid

the proba that a picked family has
kids of the same gender.

$1 - 2$

Case 1: $\frac{1}{2} \times P_k \times 2$

Case 2: $(\frac{1}{2})^2 P_k$ (2)

$\frac{1}{t} P_k$

$P_k + \frac{1}{2} P_k + \frac{1}{4} P_k + \frac{1}{8} P_k + \frac{1}{16} P_k$

$\frac{P_k}{1-2}$

$P_k (1 - 2^{n+1})$

Continuous RV's:

S. $\quad X : S \longrightarrow \mathbb{R}$

Def (of rv): X is a continuous rv if $\exists f : \mathbb{R} \to \mathbb{R}$ s.t.
$$P(a \le X \le b) = \int_a^b f(x)\,dx, \quad \forall a \le b.$$

f: density of X = probability distribut^{e} Fct^{e} (pdf).

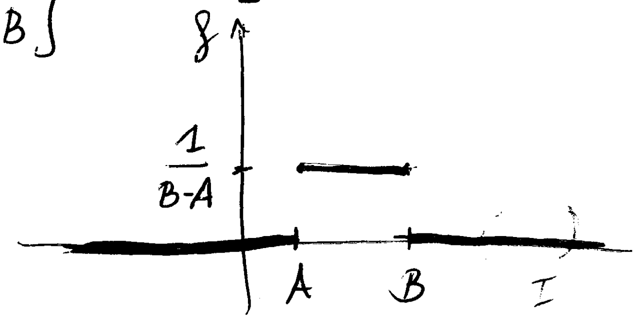
prop 1:
$f(x) \geq 0 \quad \forall x \in \mathbb{R}$

cuz $\forall a \le b$, $\underbrace{P(a \le X \le b)}_{\geq 0} = \int_a^b f(x)\,dx \geq 0 \Rightarrow f \geq 0$

prop 2: $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 1 \qquad \cancel{\left(= \int_{-\infty}^{+\infty}\right.} \Rightarrow \left(= P(-\infty \le X \le +\infty)\right) = 1$

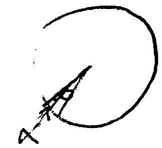
Exple: Uniform Distribut^{e} in an interval $[A, B]$

$$f(x) = \begin{cases} \dfrac{1}{B-A} & \text{if } x \in [A, B] \\ 0 & \text{otherwise} \end{cases}$$



$f(x) \geq 0$

$$\int_{\mathbb{R}} f(x)\,dx = \int_A^B \frac{1}{B-A}\,dx = \frac{1}{B-A}(B-A) = 1$$

e.g.:  the proba to reach the slice α is $\boxed{p = \frac{\alpha}{360^\circ}}$

$$P(X \in I) = 0 \qquad I \cap [A,B] = \emptyset$$

$$X : S \longrightarrow [A,B]$$

$$I = [c,d] \qquad P(X \in [c,d]) = \frac{1}{B-A}\int_c^d dx = \frac{d-c}{B-A}$$

(if $c = A$ & $d = B$ then $\frac{d-c}{d-c} = 1$)

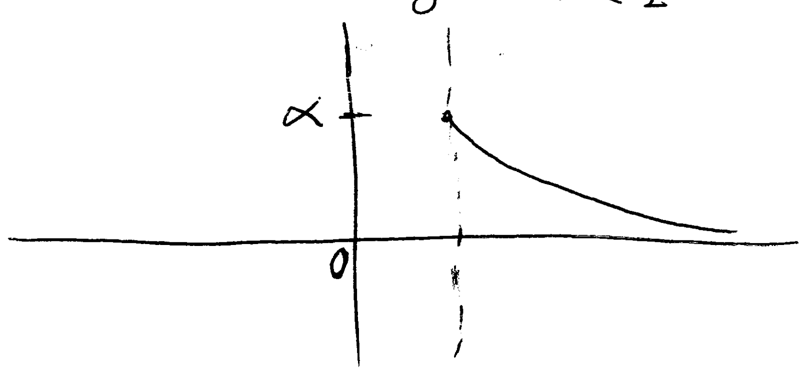
Let X be a continuous r.v.
then $\forall c \in \mathbb{R},\ P(X=c) = 0$

$$P(X=c) = P(c \le X \le c) = \int_c^c f(x)\,dx = 0.$$

(this is ≠ from the discrete case).
$\{X=c\} \neq \emptyset$ & $P(X=c)=0$ for continu

$\alpha > 0$

$$f(x) = \begin{cases} \alpha e^{-\alpha(x-1)} & \text{if } x>1 \\ 0 & \text{if } x<1 \end{cases}$$



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{+\infty}^{+\infty} \alpha e^{-\alpha(x-1)}\,dx$$

$$= \alpha e^{\alpha} \int^{+\infty} e^{-\alpha x}\,dx = \alpha e^{\alpha} \left.\frac{e^{-\alpha x}}{-\alpha}\right|_{1}^{+\infty}$$

$$= \frac{-\alpha e^{-\alpha}}{\alpha}\left[0 - e^{-\alpha}\right]$$

$$= 1$$

Let X with pdf $f(x) = \begin{cases} \alpha e^{-\alpha(x-1)} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$

$$P(X \leq 2) = \int_{-\infty}^{2} f(x)\,dx$$

$$= \int_{1}^{2} \alpha e^{-\alpha(x-1)}\,dx$$

$$= \alpha e^{\alpha} \int_{1}^{2} e^{-\alpha x}\,dx$$

$$= \frac{-\alpha}{\alpha} e^{\alpha}\left[e^{-\alpha(2)} - e^{-\alpha}\right]$$

$$= -e^{\alpha}\left[e^{-2\alpha} - e^{-\alpha}\right]$$

$$= e^{\alpha}\left(e^{-\alpha} - e^{-2\alpha}\right)$$

$$\boxed{P(X \leq 2) = 1 - e^{-\alpha}}$$

Exercice: A prof never finishes at the end of the hour & never finishes after more than 2 minutes after.

X = time between the end of the hour & the end of the c

X has proba distribut$^{\circ}$ fct$^{\circ}$ (pdf) $f(x) = \begin{cases} Kx^2 & \text{if } x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$

(1) what's the value of K?

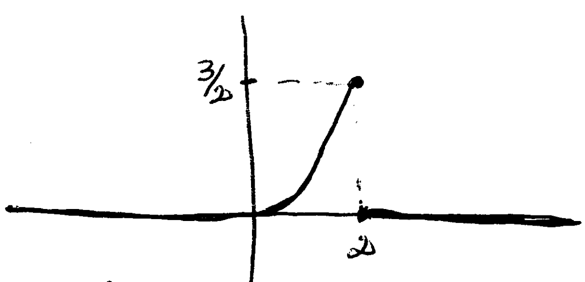
$$\int_0^2 f(x)\,dx = 1 \Rightarrow K\int_0^2 x^2\,dx = 1$$

$$\Rightarrow K\left[\frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow K\left(\frac{8}{3}\right) = 1$$

$$\Rightarrow \boxed{K = \frac{3}{8}}$$

$$f(x) = \begin{cases} \frac{3}{8}x^2 & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$



proba that he finishes within the first minute:

$$P(0 \leq X \leq 1) = \frac{3}{8}\int_0^1 x^2\,dx = \frac{1}{8}$$

$$P(60_{\text{sec}} \leq X < 90^{\text{seconds}}) = \frac{3}{8}\int_{1\,\text{min}}^{1.5\,\text{min}} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_1^{1.5} = \left(\frac{3}{8}\right)\left(\frac{1}{3}\right)(1.5^3 - 1^3) =$$

$$\frac{\binom{12}{12}}{\binom{18}{12}}$$

Midterm 1 Correct^{e}:

Exo 1:

1) $$\frac{\binom{12}{6}}{\binom{18}{6}}$$

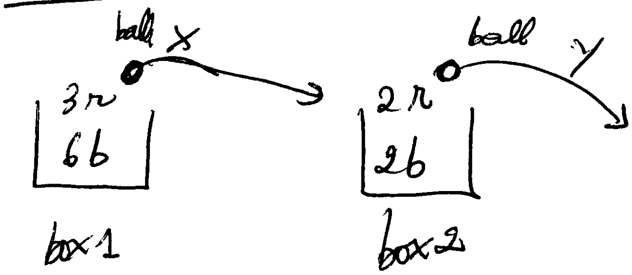
2) $$\frac{\binom{3}{2}\left[\binom{12}{6}-\binom{2}{1}\binom{6}{6}\right]}{\binom{18}{6}}$$

or

$$\frac{\binom{3}{2}\left[\binom{6}{1}\binom{6}{5}+\binom{6}{2}\binom{6}{4}+(\dots)+\binom{6}{5}\binom{6}{1}\right]}{\binom{18}{6}}$$

3) $$\frac{\binom{6}{3}\binom{6}{3}}{\binom{18}{6}}$$

Exo 2:



		P(Y=r | case)	P(case)
Case 1	$X=r$	$3/5$	$3/9$
Case 2	$X=b$	$\frac{2}{5}$	$6/9$

1) $$P(Y=r)=\left(\frac{3}{5}\times\frac{3}{9}\right)+\left(\frac{2}{5}\times\frac{6}{9}\right)=\frac{7}{15}$$

2) $$P(X=r \mid Y=r)=\frac{3/5\times\frac{3}{9}}{\frac{7}{15}}=$$

$E(X^2) = \sum x^2 p(x)$

$E(kX+b) = kE(X) + b$

$E(X_1 + X_2) = E(X_1) + E(X_2)$

$V(kX+b) = k^2 V(X)$

Exo 3:

Case	P(G|case)	P(case)
H	95%	40%
M	60%	35%
L	10%	25%

1) $P(G) = P(G|H)P(H) + P(G|M)P(M) + P(G|L)P(L)$

$P(G) = 0.615$

2) $P(H|G) = \dfrac{P(G|H)P(H)}{P(G)} = 0.618$

3) $P(H|\bar{G}) = \dfrac{P(\bar{G}|H)P(H)}{P(\bar{G})}$

$= \dfrac{(1-0.95)\times 0.4}{1-0.615}$

$P(H|\bar{G}) = 0.0519.$

Exo 4:

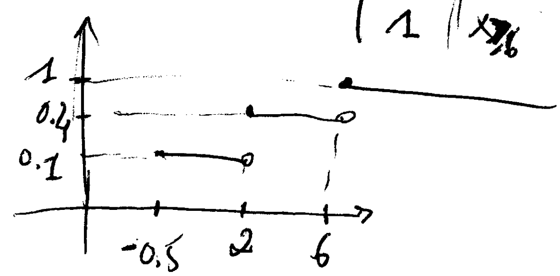
1) Pmf:

6 r
3 b
1 g

X	P
6	0.6
2	0.3
-0.5	0.1

Cdf:

$F(x) = P(X \leq x) = \begin{cases} 0 & x < \\ 0.1 & 0.5 \leq x < \\ 0.4 & 2 \leq x < \\ 1 & x \geq 6 \end{cases}$



2) $E(X) = \sum_k x_k p(x_k) = 4.15$

$E(X^2 - 5) = E(X^2) - 5$

Disc

Chap 2 review:

Random variable $X = x_k$, $k = 1, 2, \dots$

pmf: $P(X = x_k) = P(x_k)$

cdf: $F(x) = P(X \le x)$
$$= \sum_{x_k < x} P(X = x_k)$$

Expectat° $E(X)$

Variance $V(X)$

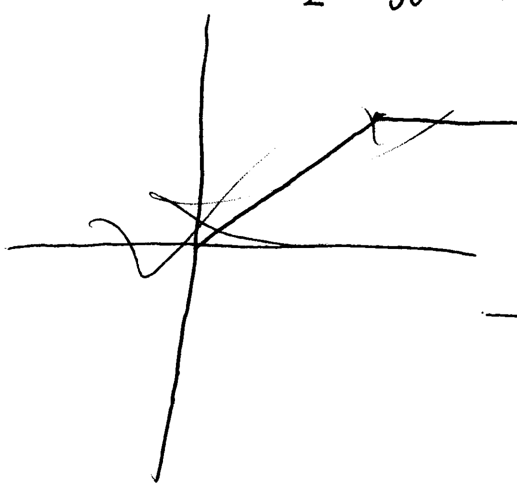
Distribut°s	Pmf ~~P(...)~~	$E(X)$	$V(X)$
Binomial	$\binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson	$\frac{e^{-\lambda} \lambda^x}{x!}$	λ	λ
ypergeometric		$n\left(\frac{M}{N}\right)$	$\frac{(N-n)}{N-1} n \frac{N}{M}\left(1-\frac{M}{N}\right)$
egative inomial		$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$

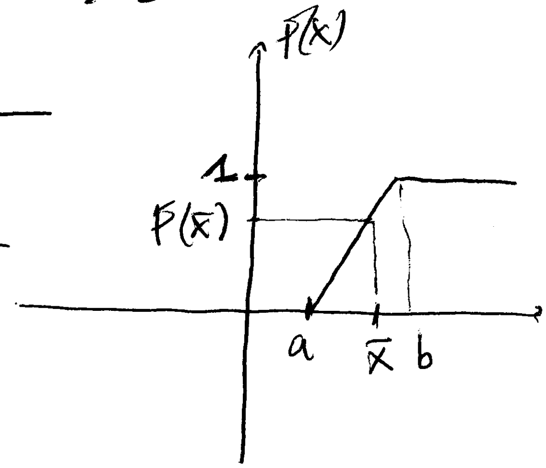
HW: ch3: ~~32~~, 38, 52, 66, 70, 78, 84

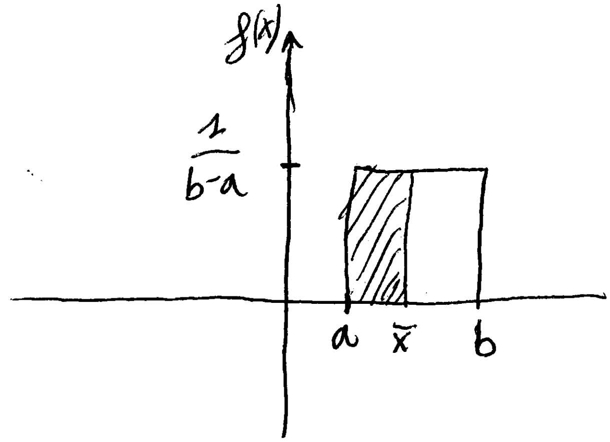
Monday ch4: 4

$$\lim_{x \to -\infty} F(x) = 0 \quad \& \quad \lim_{x \to +\infty} F(x) = 1$$

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$$







FTC: If f is differentiable, then $f(t) = F'(t)$

$$P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt.$$

$a \in \mathbb{R}$.

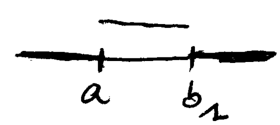
$$\mathbb{P}(X > a) = \int_{a}^{+\infty} f(t)\, dt = 1 - \mathbb{P}(X \leq a) = 1 - F(a)$$

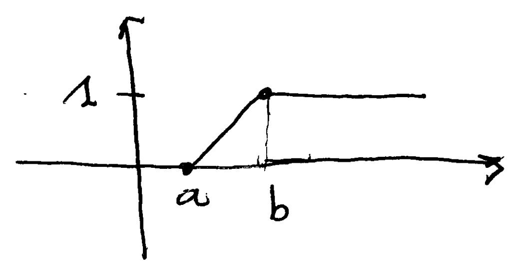
X is continuous

$F(x)$ cdf of X how to find f

$$F(x) = \int_{-\infty}^{x} f(t)\, dt \qquad \text{FTC (Fundamental Thm of Calcul}$$

$$F(x) = \begin{cases} 0 \\ \dfrac{x-a}{b-a} \\ 1 \end{cases} \Longrightarrow$$





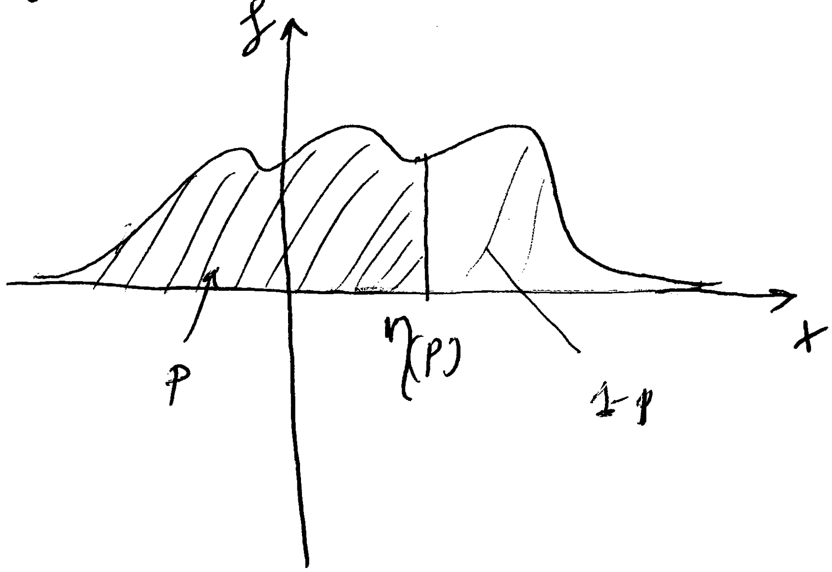
50^{th} percentile = median

<u>Percentile:</u>

$p \in (0,1)$

<u>Def.:</u> the $100p$-th percentile of the distribut° of X is denoted by $\eta(p)$ and satisfies the following relat°

$$P = \int_{-\infty}^{\eta(p)} f(t)\,dt = F(\eta(p))$$



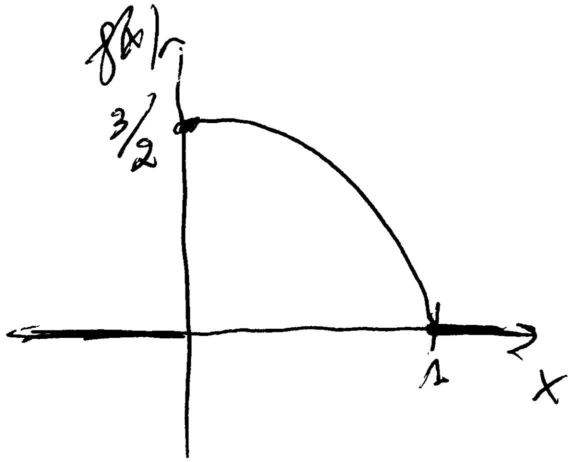
$$f(x) = \begin{cases} 3/2(1-x^2 & x \in [0,1] \\ 0 & \end{cases}$$

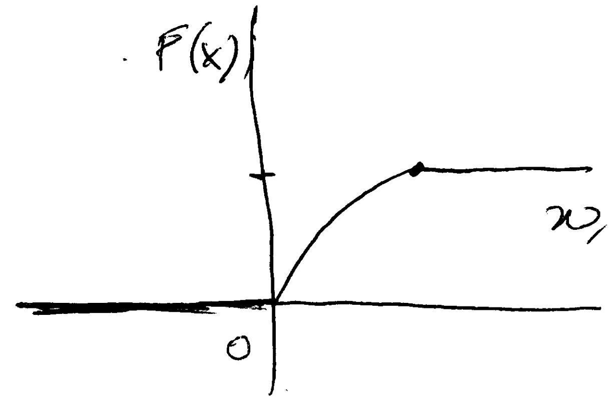
$$3/2 \int_0^1 (1-x^2)\,dx = \frac{3}{2}\left(\left[x - \frac{x^3}{3}\right]_0^1\right)$$

$$= 3/2\left(1 - \frac{1}{3}\right) = 1$$

ex:

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{2}\left(x - \frac{x^3}{3}\right) & \text{when } x \in [0,1] \\ 1 & \text{for } x \geq 1 \end{cases}$$





$p \in (0,1)$

Find $\eta(p)$ s.t. $p = F(\eta(p))$

$$= \frac{3}{2}\eta(p) - \frac{\eta(p)^3}{2}$$

$$\boxed{\eta(p)^3 - 3\eta(p) + 2p = 0.}$$

w/ $p = 0.3$

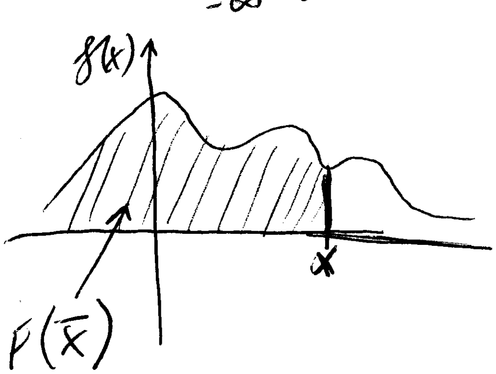
$t^3 - 3t + 0.6 = 0$

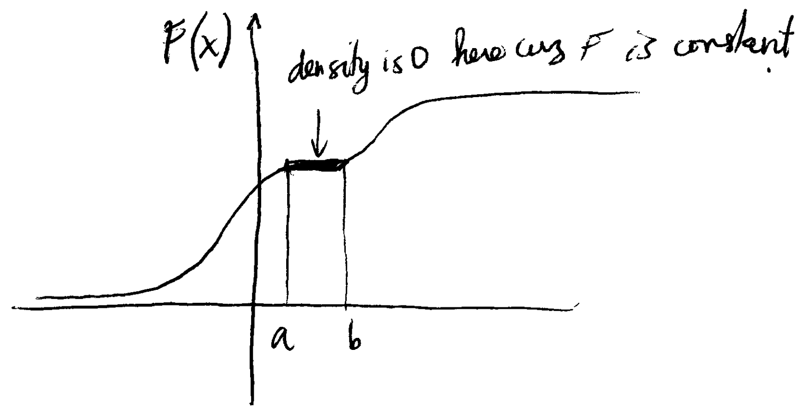
so the 30th percentile $(p = 0.3)$ is t (solut of this eq^{t}.

$X : S \longrightarrow \mathbb{R}$ r.v continuous

$$P(X \in [a,b]) = \int_a^b f(x)\,dx$$

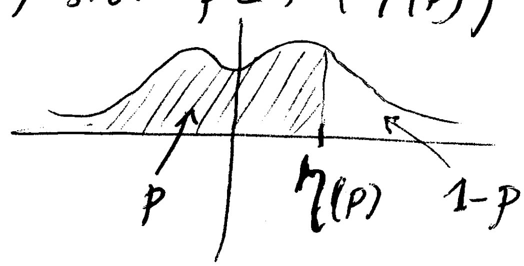
$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$





when F is constant, $f = 0$

$p \in (0,1)$ looking for $\eta(p)$ s.t. $p = F(\eta(p))$



when $p = \frac{1}{2}$ $\quad \eta(p) = \tilde{\mu} =$ median (of the solut^{e})

~~for some~~ sometimes EX is finite, sometimes it's not

Expected Value:

$$X : S \longrightarrow \{x_1, x_2, \dots, x_n\}$$

$$EX = \sum x_i \, p(x_i) = \sum x_i \, P(X = x_i)$$

$$EX =: \int_{\mathbb{R}} x f(x) dx$$

Exple:

$$f(x) = \frac{3}{2}(1 - x^2)$$

$$EX = \frac{3}{2} \int_{\mathbb{R}} x(1 - x^2) \, dx$$

$$= \frac{3}{2}\left(\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1\right)$$

$$EX = \frac{3}{8}.$$

X denotes the temperature of smthg in Celsius degrees. (continuous r.v w/ density f_X)

$Y = 32 + 1.8X$ is the temperature in ~~Fahrenh~~ Fahrenheit

let X a discrete r.v. $E(aX + b) = aEX + b$

prop: $h : \mathbb{R} \to \mathbb{R}$

X continuous r.v. w/ pdf $f(x)$. $E[h(x)] = \int_{\mathbb{R}} h(x) f(x) dx$

$\mathcal{E}\left[(X-\mu_X)^1\right]=0$

$\tilde{\mu}$

in the discrete case, $\mathcal{E}[h(X)] = \sum h(x_i)\, p(x_i)$.

Exple:

So $\mathcal{E}Y = \mathcal{E}(1.8X+32) = \int_R (1.8x+32) f_X(x)\,dx$

$= 1.8\int x f_X(x)\,dx + 32\int_R f(x)\,dx = 1.8\,\mathcal{E}X + 32.$

$\tilde{\mu}$ $\quad \frac{1}{2} = P(Y \le \tilde{\mu}_Y) = P(32+1.8X \le \tilde{\mu}_Y)$

$= P\left(X \le \frac{\tilde{\mu}_Y - 32}{1.8}\right)$

Therefore $\tilde{\mu}_X = \frac{\tilde{\mu}_Y - 32}{1.8} \Rightarrow \tilde{\mu}_Y = 32 + 1.8\,\tilde{\mu}_X$

Ex for general 100p th percentile.

$\eta_X(p) \longrightarrow \eta_Y(p)$

Def.: $V(X) = \int_R (x-\mu_X)^2 f(x)\,dx = \mathbb{E}\left[(X-\mu_X)^2\right]$

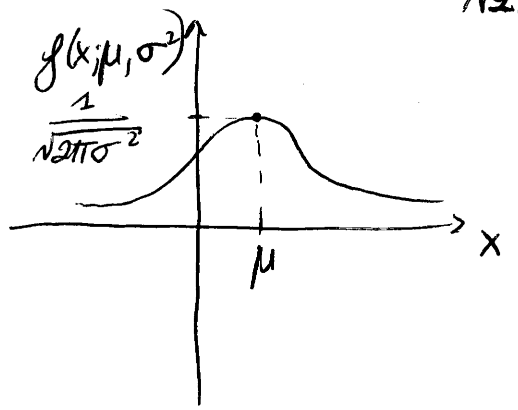
$= \int_R x^2 f(x)\,dx - \mu_X^2$

$V(X) = \sigma_X^2 \qquad \sqrt{V(X)} = \sigma_X$ = standard deviat°.

Normal Distribut^{e}:

Def.: A continuous r.v X has normal distribut^{e} of parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ if

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



• $f \geq 0$

• $\int f(x)dx = 1$

$\forall \mu \in \mathbb{R}$ & $\forall \sigma^2 > 0$,

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{-\frac{(x-\mu)}{2\sigma^2}} dx = 1$$

for $\mu = 0$ & $\sigma = 1$, we have

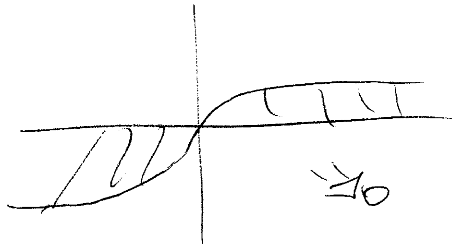
$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = 1$$

Assume that $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = 1$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

take $y = \frac{x-\mu}{\sigma}$ $dy = \frac{1}{\sigma} dx$

$dx = \sigma dy$



- program - committee
- ASPAC

If X has density $f(x;\mu,\sigma^2)$, $X \sim N(\mu,\sigma^2)$

Prop: If $X \sim N(\mu;\sigma^2)$

① $EX = \mu$

② $V(X) = \sigma^2$

Facts: ① $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}} x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \mu$

② $\frac{1}{\sqrt{2\pi\sigma^2}}\int x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sigma^2 + \mu^2$

③ For $\mu = 0$ and $\sigma^2 = 1$

$$EX = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x e^{-x^2/2}\,dx = 0$$

6.2 : 2, 10, 13
6.3 : 4
6.4 : 1, 3, 9

Discuss

Ch 4: Continuous Random Variable

R.V

Discrete	Continuous
pmf	pdf
$p(x_k) = P(X = x_k)$	$f(x) = \frac{dF(x)}{dx}$
cdf $F(x) = \sum_{x_k \le x} p(x_k)$	$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$
Expectation $EX = \sum x\,p(x)$	$EX = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\sum_k P(x_k) = 1 \quad , \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$P(a \le x \le b) = \int_a^b f(x)\,dx.$$

Exple 1: known PDF of x

$$f(x) = \begin{cases} ax+b & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

& $P(2 < x < 3) = 2P(1 < x < 2)$

Find constants a & b.

Solut:

- $\int_2^3 f(x)\,dx = 2\int_1^2 f(x)\,dx$

$\hookrightarrow \frac{5}{2}a + b = 2\left(\frac{3}{2}a + b\right) \Rightarrow \boxed{a = -2b}$ ①

- $\int_1^3 f(x)\,dx = 1$

$\hookrightarrow \int_1^3 ax+b = 1$

$\hookrightarrow \left[\frac{ax^2}{2} + bx\right]_1^3 = 1$

$\hookrightarrow \boxed{\cancel{4}\,a + 2b = 1}$ ②

① & ② $\Rightarrow a = \frac{1}{3}$ & $b = -\frac{1}{6}$

Special Distributⁿ:

→ Uniform distributⁿ:

Unif (a, b)

PDF

$$f(x) = \frac{1}{b-a} \quad a \le x \le b$$

$$E(x) = \frac{a+b}{2}$$

$$V(x) = \frac{(b-a)^2}{12}$$

CDF

$$F(x) = \frac{x-a}{b-a}$$

for $a \le x$.

→ Normal distributⁿ:

Norm (μ, σ^2)

PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma}}$$

$$E(x) = \mu$$

$$V(x) = \sigma^2$$

For $\mu = 0$, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 1$

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{1}{2}$$

Exple 2: R.V $X \sim$ Unif $[2,5]$. We have 3 independent observat° of X. Find the probability that at least 2 observat° are bigger than 3.

Solut°: $P(2 < X \leq 3) = \frac{3-2}{5-2} = \frac{1}{3}$ & $P(X > 3) = \frac{2}{3}$

$P(\text{at least two}) = P(\text{two}) + P(\text{three}) = 3\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 = \frac{20}{27}$

$\binom{3}{2}$

(Binomial dist. cuz either "bigger than 3" or "not b. than 3")

Exple 3: Telecommunicat° transfer "0" & "1" 512×10^3 per s. There could be an error because of the distribut°. The error rate is 10^{-7}. Find the proba. that 1 error occurs in 10s.

Solut°:

X: number of errors $\lambda = np = 512 \times 10^4 \times 10^{-7} = 0.512$

$p(X=1) = b(1; 512 \times 10^4, 10^{-7}) \approx p(1, 0.512)$

$= \frac{0.512}{1!} e^{-0.512} \approx 0.3$

HW 5:

- as bigger is σ, as more spread is the graph of the normal distribut^{e}.
- $N(\mu, \sigma^2)$ = Normal distribut^{e} w/ parameters μ & σ

$\mu \in \mathbb{R}$, $\sigma^2 > 0$

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$f > 0$

$$\int_{\mathbb{R}} f(x, \mu, \sigma^2)\, dx = 1$$

Normal Distribut^{e} of parameter μ & σ^2

$X : S \to \mathbb{R}$ r.v. has Normal distribut^{e} if its densi is $f(x, \mu, \sigma^2)$.

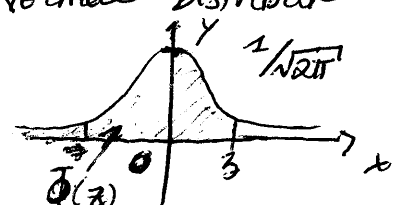
If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$. & $Var(X) = \sigma^2$

Assume this particular case that $\mu = 0$, $\sigma^2 = 1$

$$f(x; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

$N(0,1)$ is standard Normal Distribut^{e}

In this case, we have:



Let $Z \sim N(0,1)$

(cdf →) $F(z) = \mathbb{P}(Z \leq z) = \Phi(z)$

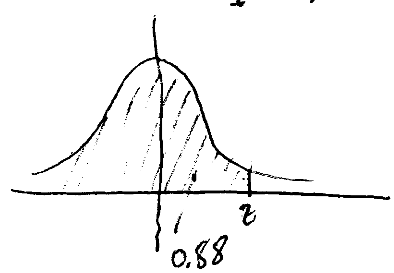
① $\mathbb{P}(Z > z) = 1 - P(Z \leq z)$
$= 1 - \Phi(z)$

$\Phi(-z) = \mathbb{P}(Z > z)$ (by symmetry)
$= \mathbb{P}(Z > z)$
$= 1 - \Phi(z)$

$\Phi(-z) = 1 - \Phi(z) \quad \forall x \in \mathbb{R}$

here Z is a standard normal distribu

Ex. Compute the 88th percent of Z. Find z s.t.
$0.88 = \Phi(z)$



so to table $\phi(1.17) = 0.879$
$\phi(1.18) = 0.881$

so by interpolat°, the right value is $1.175 = z$

so $P(z \leq 1.175) = 88\%$.

What is the value of the 12th percentile?
$z = -1.175$

$\cancel{P(z}$

$P(\cancel{z) = P(z \leq z}$

$F(z) = P(z < z) = \Phi(z)$

$X \sim N(\mu, \sigma^2)$ $\qquad f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$

$Z \sim N(0, 1)$

$$Y = \frac{X-\mu}{\sigma}$$

Propriété : $Y \sim N(0, 1)$

Preuve : $\forall z \quad \underline{\mathbb{P}(Y \leq z) = \Phi(z)}$

$$P(Y \leq z) = P\left(\frac{X-\mu}{\sigma} \leq z\right) = P(X \leq \sigma z + \mu)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\sigma z + \mu} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$y = \frac{x-\mu}{\sigma} \Rightarrow dy = \frac{1}{\sigma} dx$$

$$\text{so } P(X \leq \sigma z + \mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{z} e^{-y^2/2} \sigma \, dy$$

$$= \Phi(z)$$

$X \sim N(\mu, \sigma^2)$

$X \sim \sigma Z + \mu \qquad Z \sim N(0, 1)$

$$\Phi(-z) = 1 - \Phi(z)$$

$$P(a \le X \le b) = P(a \le \sigma Z + \mu \le b)$$
$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$\Phi = \Phi$$

exo (35) in the book:

(1) the proba that the diameter will be at least 10 inches

X = diameter

$X \sim N(8.8, 2.8^2)$.

$$P(X \ge 10) = 1 - P(X \le 10)$$

$$X = 2.8 Z + 8.8$$

so $P(X \le 10) = P(2.8 Z + 8.8 \le 10)$

so $P(X \ge 10) = 1 - P(2.8 Z + 8.8 \le 10)$

$$= 1 - P\left(Z \le \frac{1.2}{2.8}\right)$$

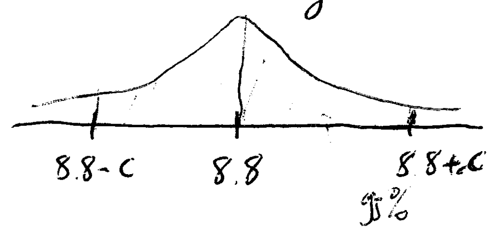
$$P(X \ge 10) = 1 - \Phi\left(\frac{3}{7}\right)$$ (go to book)

$$P(5 \le X \le 10) = P\left(\frac{5-8.8}{2.8} \le Z \le \frac{10-8.8}{2.8}\right)$$
$$= \Phi\left(\frac{1.2}{2.8}\right) - \Phi\left(\frac{-3.8}{2.8}\right)$$
$$= \Phi\left(\frac{1.2}{2.8}\right) - \left(1 - \Phi\left(\frac{+3.8}{2.8}\right)\right)$$

table p. 669

② what's the value of $c > 0$ s.t. the interval $(8.8-c, 8.8+c)$ includes the 98% of all diameters

Solut^{k}.



c s.t. $P(8.8 - c \leq X \leq 8.8 + c) = 0.98$

$X \sim 2.8Z + 8.8$

$$P(8.8 - c \leq X \leq 8.8 + c) = P\left(\frac{-c}{2.8} \leq Z \leq \frac{c}{2.8}\right)$$

$$= \phi\left(\frac{c}{2.8}\right) - \phi\left(\frac{-c}{2.8}\right)$$

$$= \phi\left(\frac{c}{2.8}\right) - 1 + \phi\left(\frac{c}{2.8}\right)$$

$$= 2 \cdot \phi\left(\frac{c}{2.8}\right) - 1$$

Fix $c > 0$ s.t. $0.98 = 2\phi\left(\frac{c}{2.8}\right) - 1 \Rightarrow \phi\left(\frac{c}{2.8}\right) = 0.99$

$\frac{c}{2.8} = 2.33$

so $c = (2.33)($

$c =$

③ Proba that at least 1 diameter exceeds 10 inches?:

Let A this event

$P(A) = 1 - P(A^c) = 1 - [P(X \leq 10)]^4$ (as there are 4 t

$A^c = \{$no tree exceeds 10 inches$\}$

$p \in (0,1)$ $\quad X \sim N(\mu, \sigma^2)$

$$\frac{X-\mu}{\sigma} \sim N(0,1).$$

$F(x) = P(X \leq x)$

Find $\eta_p^X \in \mathbb{R}$ st. $F(\eta_p^X) = p$.

$$p = \mathbb{P}(X \leq \eta_p^X) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \eta^X_{\frac{p-\mu}{\sigma}}\right)$$

$$= \mathbb{P}\left(Z \leq \eta^X_{\frac{p-\mu}{\sigma}}\right) \Rightarrow \eta_p^Z = \eta^X_{\frac{p-\mu}{\sigma}}$$

$$\boxed{\eta_p^X = \sigma \eta_p^Z + \mu}$$

Exponential Distribut^e:

$\lambda > 0$ parameter

Def: X has exponential Distribut^e of parameter λ ($\lambda > 0$) if its pdf is given by $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{\mathbb{R}} f(x;\lambda)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{+\infty}$$

$$= \lim_{M \to +\infty} -\underbrace{e^{-\lambda M}}_{0} + 1 = \underline{\underline{1}}$$

⚠ the integral of a positive fct° SHOULD be positive ⚠.

If $X \sim \mathrm{Exp}(\lambda)$ $\quad \mathbb{E}X = \frac{1}{\lambda}$ $\quad Var(X) = \frac{1}{\lambda^2}$. $\quad \sigma_X = \frac{1}{\lambda}$.

$$\mathbb{E}X = \int_{\mathbb{R}} x f(x,\lambda)\,dx = \int_0^{+\infty} x \underbrace{\lambda e^{-\lambda x} dx}_{\frac{d}{dx}(-e^{-\lambda x})}$$

$$= -xe^{-\lambda x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}\,dx$$

$$= \lim_{M \to +\infty} \left[-Me^{-\lambda M}\right] + \int_0^{+\infty} e^{-\lambda x}\,dx = \int_0^{+\infty} e^{-\lambda x}\,dx$$

$$= \frac{\lambda}{\lambda} \int_0^{+\infty} e^{-\lambda x}\,dx \rightsquigarrow 1$$

$$= \boxed{\frac{1}{\lambda}}.$$

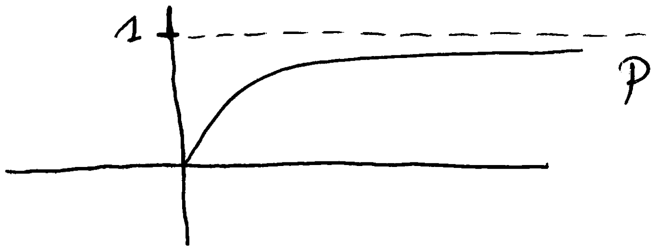
$X \sim \mathrm{Exp}(\lambda)$ $\quad \lambda > 0$ $\quad t_0 > 0$

$$\mathbb{P}(X > t_0 + t \mid X > t_0) = \frac{\mathbb{P}(X > t_0 + t,\ X > t_0)}{\mathbb{P}(X > t_0)} = \frac{\mathbb{P}(X > t_+}{\mathbb{P}(X:}$$

Let $F(x) = \mathbb{P}(X \leq x)$.

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \lambda e^{-\lambda t}\,dt & x > 0 \end{cases}$$

$$\quad \| \ 1 - e^{-\lambda x}$$



$x \geq 0$

$P(X > x) = 1 - F(x) = e^{-\lambda x}$

$$\frac{P(X>t+t_0)}{P(X>t_0)} = \frac{e^{-\lambda(t+t_0)}}{e^{-\lambda t_0}} = e^{-\lambda t} = P(X>t)$$

⚠ Fact: If the # of events occurring in a time interval of length t is distributed w/ a poisson distribut^e of parameter αt. and if the # of events occurring on two non overlapping time intervals are independent, then the time b/ 2 ~~successes~~ successive events is described by an exponential of parameter α.

N_t = # of events in a time interval of size t,

$$P(N_t = k) = e^{-\alpha t}\frac{(\alpha t)^k}{k!} \quad \text{(poisson distribut}^n\text{)}$$

let X, the time between 2 successive events.

$X \sim Exp(\alpha)$

$$\begin{aligned} P(X \leq t) &= 1 - P(X > t) = 1 - P(\text{no events in time } (0,t)) \\ &= 1 - e^{-\alpha t}\frac{(\alpha t)^0}{0!} \\ &= 1 - e^{-\alpha t} \end{aligned}$$