MATH 410 F09

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Notat:

- $\exists$ means there exist
- $\forall$ = for all
- $\Rightarrow$ = implies
- $\Leftrightarrow$ = equivalent to, i.e $\underset{\text{if}}{\Rightarrow}$ & $\underset{\text{only if}}{\Leftarrow}$
- $\therefore$ = therefore
- $\in$ = belongs to or in
- $\notin$ = doesn't belong

Abbreviations.

- and = &
- function = fn
- such that = s.t.
- point = pt
- iff = if & only if, $\Leftrightarrow$

Stating statements / results:

Lemma — a technical result used in order to prove either a theorem or propos.

Prop - Proposition — 'less major' result

Thm - Theorem — major result of a given section

Ex - Example

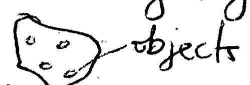
Remark

Def. - Definition (to define) a starting pt.

$\mathcal{E}$.

Sets & Fund$^{\text{as}}$:

- A set in a nutshell is a collect° of objects, called elem



- $\emptyset$ is the empty set, set with no elmts.
- $A \subseteq B$ means: "$\forall x, x \in A \Rightarrow x \in B$".

- $A = B$ means that A & B have the same elmt.
- $\underset{\text{intersect}}{A \cap B} = \{x \mid x \in A \ \& \ x \in B\}$

  set of ~~such~~ these — such that — condit° holds

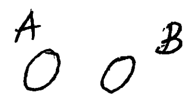
- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

  union

- $A \setminus B = \{x \mid x \in A \ \& \ x \notin B\}$; A minus B, set-theoretic

  removing B

- $\bigcup_{i=1}^{n} A_i = \{x \mid \exists\, i \text{ with } 1 \leq i \leq n \text{ s.t. } x \in A_i\}$
- A & B are disjoint if $A \cap B = \emptyset$



- $A \times B = \{\text{all pairs } (a,b) \mid a \in A, b \in B\}$ Cartesian prod

  e.g. $\{0,1\} \times \{0,1\} = \{(0,0),(1,1),(0,1),(1,0)\}$

- $\{x\}$ set with one element.

im(f) - image of f

<u>Definition</u>. (fct^e as a rule)

A fct^e consists of 3 things:

- 2 sets, called the <u>domain</u> & the <u>range</u>

& - a rule that associates to any elmt in the domain exactly one element in the range.

$$\underbrace{f}_{\text{fct}^e} : \underbrace{A}_{\text{Domain}} \longrightarrow \underbrace{B}_{\text{Range}}$$

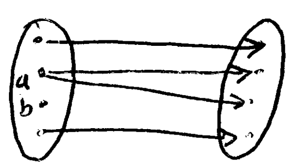
<u>Def</u>.: The set of all values of f is called its <u>image</u>

$y \in im(f)$ if $\exists x \in A$ s.t. $f(x) = y$.

the image of A is $f(A)$.

- A fct^e should assign a value to every pt in the domai (everywhere defined)
- The value in the range should be unique (well-defin

<u>Ex</u>.

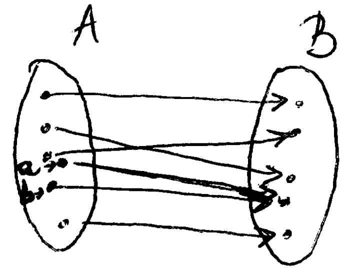


not a fct^e cuz
1) not well-defined at a
2) not defined at b.

<u>Def</u>.: $\boxed{}$ A fct^e $f: A \to B$ is <u>onto</u> (or <u>subjective</u>) if
$\forall y \in B \ \exists x \in A$ s.t. $f(x) = y$

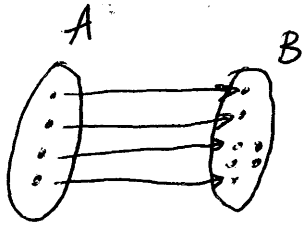
$\boxed{}$ A fct $f: A \to B$ is one-to-one (or <u>injective</u>) if
$\forall y \in B \ \exists$ at most one $x \in A$ s.t. $f(x) = y$.

Ex:



- not one-to-one (1-1) cuz a & b h

- onto cuz every pt in B is the imag a pt in A.



- not onto
- 1-1.

Ex: "Social S. number of": Americans $\longrightarrow$ 9-digit numbers

1-1 ✓

not onto (yet)

Def. f is invertible (or bijective) if it is both onto & 1-1
The inverse fct^{n} of f is denoted f^{-1}.

The Real Numbers:

standard notat^{n}:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ set of natural numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ integers

$\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}\}$ rational numbers

$\mathbb{R}$? set of real numbers

Aim: characterize $\mathbb{R}$ "systematically", then go back & characterize $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{Z}$ similarly.

03/09 The Real Numbers:

Field Axioms. The set of real # $\mathbb{R}$ is a field. That is, $\mathbb{R}$ has at least 2 elmts & there are 2 binary operat^{es}:

- addition "+" : $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$
- multiplicat^{k} "$\cdot$" $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$,

so that.

$+$
1. $+$ is commutative : $\forall x, y \in \mathbb{R}$, we have $x+y=y+x$
2. $+$ is associative: $\forall x, y, z \in \mathbb{R}$, we have $(x+y)+z = x+(y+z)$
3. $\exists$ identity elmt 0 for $+$: $\exists 0 \in \mathbb{R}$ s.t. $x+0=x$
4. $\forall x \in \mathbb{R}$ $\exists$ an additive inverse elmt $(-x)$ s.t. $x+(-x)=0$

$\cdot$
5. $\cdot$ is commutative: $\forall x, y \in \mathbb{R}$, we have $x \cdot y = y \cdot x$
6. $\cdot$ is associative. $\forall x, y, z \in \mathbb{R}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
7. $\exists$ identity for $\cdot$ $\exists 1 \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}$, we have $1 \cdot x = x$
8. $\forall x \in \mathbb{R} \setminus \{0\}$ a multiplicative inverse elmt x^{-1} s.t. $x \cdot x^{-1} = 1$

9. $\cdot$ is (left) distributive over $+$: $\forall c, x, y \in \mathbb{R}$ have
$c \cdot (x+y) = c \cdot x + c \cdot y$

10. $\cdot$ is right distributive over $+$: $(x+y) \cdot c = x \cdot c + y \cdot c$ $\leftrightarrow$ proof using an

⚠ DO NOT USE INTUITION
⚠ JUST USE WHATEVER IS GIVEN AS PROPOSITION

<u>Theorem 1</u>: The following are true in $\mathbb{R}$:

1. $\forall x \in \mathbb{R}$, we have $0x = 0$
2. $0 \neq 1$
3. Additive inverses are uniq, i.e. if $x \in \mathbb{R}$ & x' & $\bar{x}$ both have property ④, then $x' = \bar{x}$
4. $\forall x \in \mathbb{R}$, we have $(-1)x = -x$

<u>Proof</u>:

1. Let $x \in \mathbb{R}$. Then $0x = (0+0)x$ by ③
$= x(0+0)$ by ⑤
$= x0 + x0$ by ⑨
(*) $0x = 0x + 0x$ by ⑤ (commutative)

Now $0 = 0x + (-0x)$ by ④
$= 0x + 0x + (-0x)$ by (*)
$= 0x + (0x + (-0x))$ by ② (associativity)
$0 = 0x + 0$ by ④
$0 = 0x$ by ③ (identity elmt)

2. Since $\mathbb{R}$ has at least 2 elmts, $\exists$ an $x \in \mathbb{R} \setminus \{0\}$
Now suppose for a contradict° that $0 = 1$.
Then, $x = 1 \cdot x$ by ⑦
$= 0 \cdot x$ by assumpt°
$= 0$
Contradict° ~~to~~ $x \in \mathbb{R} \setminus \{0\}$. Therefore, $0 \neq 1$.

3. Note that if x' & $\bar{x}$ both have property ④, then

$$
\begin{aligned}
x' &= x' + 0 && \text{by ③} \quad \text{introducing 0 cuz we want to Cal } x+\bar{x}. \\
&= x' + (x + \bar{x}) && \text{by ④ \& assumpt}^{n} \\
&= (x' + x) + \bar{x} && \text{by ② (associativity)} \\
&= (x + x') + \bar{x} && \text{by ① (commutativity)} \\
&= 0 + \bar{x} && \text{by ④ and assumpt}^{\circ} \\
x' &= \bar{x} + 0 && \text{by commutativity (②).} \\
x' &= \bar{x} && \text{by ③.}
\end{aligned}
$$

4. Note that

$$
\begin{aligned}
x + (-1)x &= 1 \cdot x + (-1)x && \text{by ⑦ (Identity for mu} \\
&= (1 + (-1)) \cdot x && \text{by ⑩ (right distribution)} \\
&= 0 \cdot x && \text{by ④} \\
&= 0 && \text{by part 1 of theorem}
\end{aligned}
$$

Part 3 of theorem $\Rightarrow x + \boxed{(-1)x} = 0$
is ↳ The additive inverse $(-x)$.

<u>The Order Axiom</u>: The real # $\mathbb{R}$ contain a subset $\mathbb{R}^+ = \mathcal{P}$ called the positive real numbers s.t.

$0 \notin \mathbb{R}^+$

① $\forall x, y \in \mathbb{R}^+$, we have $x+y \in \mathbb{R}^+$ & $xy \in \mathbb{R}^+$

② $\forall x \in \mathbb{R}$, exactly one of the following holds: $\begin{cases} \text{Either } x \in \mathbb{R}^+ \\ \text{or } -x \in \mathbb{R}^+ \\ \text{or } x = 0 \end{cases}$

$\mathbb{R}^+$ is closed under addit^{n} & multiplicat^{n} cuz $\underset{\text{in } \mathbb{R}^+}{a+b} \in \mathbb{R}^+$ & $\underset{\text{in } \mathbb{R}^+}{ab} \in \mathbb{R}$

. A real number x is called <u>negative</u> ~~to~~ iff (if & only if) $-x \in \mathbb{R}^+$

. here on:

$y + (-x) = y - x$ difference of y & x

Binary operat$^{\underline{n}}$ - called <u>substract$^{\underline{n}}$</u>.

<u>Def.</u> 4: (1) For $x, y \in \mathbb{R}$, we say x <u>less</u> than y, $x < y$, iff $y - x \in \mathbb{R}^+$

(2) " " " " " or equal, $x \leq y$ iff $x < y$ or $x = y$

(3) " we say x <u>greater</u> than y, $x > y$ iff $x - y \in \mathbb{R}$

(4) " " " or equal to, $x \geq y$, iff $y \leq x$

Prop$^{\underline{n}}$: $\leq$ is an order relat$^{\underline{n}}$ on $\mathbb{R}$. That is

i) "$\leq$" is reflexive ; $\forall x \in \mathbb{R}$, have $x \leq x$

ii) "$\leq$" is symmetric ; $\forall x, y \in \mathbb{R}$, have $x \leq y$ & $y \leq x \Rightarrow x = y$

iii) "$\leq$" is transitive, $\forall x, y, z \in \mathbb{R}$, have $x \leq y$ & $y \leq z \Rightarrow x \leq z$

Moreover, the relat$^{\underline{n}}$ "$\leq$" is a total order relat$^{\underline{n}}$, i.e. for any $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$.

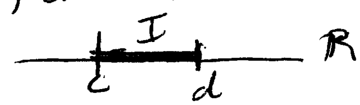
<u>Proof</u>: i) Immediate since includes equality $x \leq x$

ii) Let $x \le y$ & $y \le x$ & suppose for a contradict^n that $x \ne y$
then $x-y \in \mathbb{R}^+$ & $-(x-y) = y-x \in \mathbb{R}^+$,
cannot be by Order Axiom ②
thus $\le$ must be symmetric.

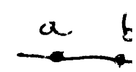
iii) Let $x \le y$ & $y \le z$.
Nothing to prove if one of inequalities is equality.
Thus assume $x<y$ & $y<z$, means: $y-x \in \mathbb{R}^+$
& $z-y \in \mathbb{R}^+$
$(z-y)+(y-x) = z-x \in \mathbb{R}$
by Order Axiom ①

$\Rightarrow \quad x<z.$

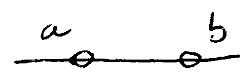
09/08/09 • An Interval is a set $I \subseteq \mathbb{R}$ so that $\forall\, c, d \in I$ & $x \in$
the inequalities $c<x<d$ imply $x \in I$



closed (1) $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ closed interval



open (2) $(a,b) = \{x \in \mathbb{R} \mid a<x<b\}$



$(a,\infty) = \{x \in \mathbb{R} \mid a < x\}$
$(-\infty, b) = \{x \in \mathbb{R} \mid x<b\}$
$(-\infty, \infty) = \mathbb{R}$, $\pm\infty \notin \mathbb{R}$

half open (3) $[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$
$[0, \infty)$
$(-\infty, a]$

Theorem: Properties of order relat$^=$. Let $x, y, z \in \mathbb{R}$

(1) The number x is positive iff $x > 0$ & x is negative iff x.

(2) If $x \le y$ then $x+z \le y+z$

(3) If $x \le y$ & $z > 0$ then $xz \le yz$

(4) If $x \le y$ & $z < 0$, then $xz \geq yz$

(5) If $0 < x \le y$ then $y^{-1} \le x^{-1}$

Proof: (3) Let $x \le y$ & $z > 0$ then $y - x \in \mathbb{R}^+$ (ii) or $y = x$ (i)

(i) If $y = x$ we get $yz = xz$ & this in particular ~~this satisfies~~ $xz \le yz$

(ii) $y - x \in \mathbb{R}^+$ & $z > 0$ means $z \in \mathbb{R}^+$

so $yz - xz = (y-x)z \in \mathbb{R}^+$ cuz $\mathbb{R}^+$ is closed under multiplicat

By definit$^=$, this implies that $xz < yz$ & in particular $xz \le$

(5) Nothing to prove (it's obvious) if $x = y$

Assume $x < y$. Suppose for a contradict$^=$ that $x^{-1} < y^{-1}$.

Then, by part (3), $1 = x^{-1}x < y^{-1}x$

Hence

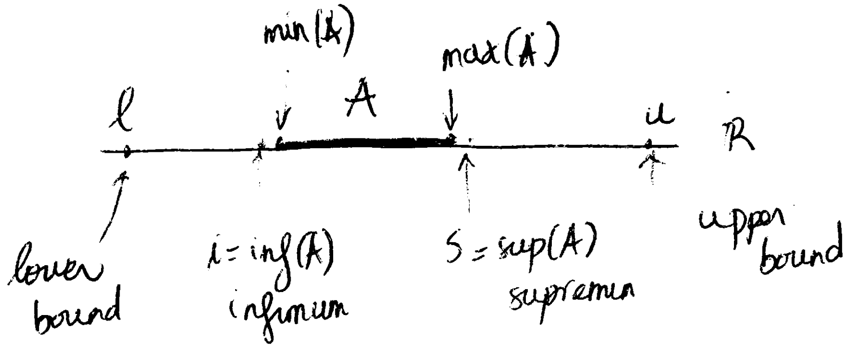
$$x < y \cdot 1 < y \cdot \underbrace{y^{-1}x}_{\text{greater than } 1} = x$$

So we got $x < x$, contradict$^=$

Assumpt^e wrong

Use it's inverse, which is $y^{-1} \le x^{-1}$

■



<u>Proposit² 1</u>: Suprema are uniq. That is, if the set $A \subseteq \mathbb{R}$ is bounded above $s, t. \in \mathbb{R}$ both are suprema of A, then $s = t$.

proof. Let $A \subseteq \mathbb{R}$ & $s, t \in \mathbb{R}$ be as indicated.

Then s is an upper bound of A & because t is a supremum of A we get $s \geq t$.

Similarly, t is an upper bound of A & because S is a sup. of A, get $t \geq s$. Therefore $s = t$.

<u>Completeness Axiom</u>: Every nonempty subset S of $\mathbb{R}$ that has an upper bound has a lowest upper bound.

<u>Proposit² 2</u>: Let $S \subset \mathbb{R}$ be a nonempty subset of $\mathbb{R}$ that is bounded above & let $s = \sup(S)$.

then $\forall \varepsilon > 0$, $\exists$ an elmt $x \in S$ s.t. $s - x < \varepsilon$

~~proof~~ Suppose for a contradict² that $\exists\ \varepsilon > 0$ so that $\forall x \in$ we have $s - x \geq \varepsilon$

Then $\forall x \in S$ we would obtain $s - \varepsilon \geq x$,

i.e $s - \varepsilon$ would be an upper bound of S. contradicts that

$s - \varepsilon < s$. s is the lowest upper bound.

so $s - x < \varepsilon$.

The natural numbers.

Thm/Def 1: there is a subset $\mathbb{N} \subseteq \mathbb{R}$, called the natural nu so that

(1) $1 \in \mathbb{N}$

$\mathbb{N}$ (2) For each $n \in \mathbb{N}$, the number $n+1$ is also in $\mathbb{N}$

(3) Principle of induct^o: If $S \subseteq \mathbb{N}$ is such that $1 \in$ S for each $n \in S$ we also have $n+1 \in S$, then $S = \mathbb{N}$

proof: call a subset $A \subseteq \mathbb{R}$ an inductive set iff $1 \in A$ $\forall\ a \in A$ we also have $a+1 \in A$.

(1) Let $\mathbb{N}$ be the set of all elmts of $\mathbb{R}$ that are in all inductive sets. Because 1 is an elmt of every indu set, we have $1 \in \mathbb{N}$.

(2) If $n \in \mathbb{N}$, then n is in every inductive set, which means $n+1$ is in every inductive set, & hence $n+1 \in \mathbb{N}$

(3) Because the elmts of $\mathbb{N}$ are contained in all inductive sets, we get $\mathbb{N} \subseteq S$ $\Rightarrow$ $S = \mathbb{N}$

Theorem 2 (principle of Induct^e) Let $P(n)$ be a statement abou the natural number n.

If $P(1)$ is true & if $\forall n \in \mathbb{N}$ truth of $P(n)$ implies $P(n+1)$ then $P(n)$ holds for all natural numbers

proof: Let P be as indicated, & consider the set $S = \{n \in \mathbb{N} \,/\, P(n) \text{ is true}\}$

then $1 \in S$ (given)

For every ~~$\forall$~~ $n \in S$ the statement $P(n)$ is true, hence $p(n+1)$ is true, which means $n+1 \in S$

By theorem 1, have $S = N$ & so $P(n)$ is true $\forall n \in N$.

"$1 \in S$" base step

"$n \in S \Rightarrow n+1 \in S$" induct^e step.

Proposit^e 3 The natural #s are closed under addit^e & multiplicat^e.

If $m, n \in \mathbb{N}$, then $m+n \in \mathbb{N}$ & $mn \in \mathbb{N}$

proof: By induct^e.

- Let $m \in \mathbb{N}$ arbitrary, let $S_m = \{n \in \mathbb{N} \mid m+n \in \mathbb{N}\}$

then $m \in \mathbb{N} \Rightarrow m+1 \in \mathbb{N}$ & hence $1 \in S_m$

Moreover, if $n \in S_m$ then $m+n \in \mathbb{N}$, & hence

$m+(n+1) = (m+n)+1 \in \mathbb{N}$, i.e. $n+1 \in S_m$.

"Induct^e" $\Rightarrow S_m = \mathbb{N}$

, mn similar.

Proposit^e 4: Let $m, n \in \mathbb{N}$ be such that $m > n$.

Then $m - n \in \mathbb{N}$

proof: $m \in \mathbb{N} \Rightarrow \begin{cases} m-1 \in \mathbb{N} \\ \text{or} \\ m-1 = 0 \end{cases}$

$A = \{m \in \mathbb{N} \mid m-1 \in \mathbb{N} \text{ or } m-1 = 0\}$

then $1 \in A$ & if $m \in A$ then $(m+1) - 1 = m \in A \subseteq \mathbb{N}$

so $m+1 \in A$

Hence $A = \mathbb{N}$ Induct^e.

• Now we let

$$S = \{ n \in \mathbb{N} \mid (\forall m \in \mathbb{N} \mid m > n \Rightarrow m - n \in \mathbb{N}) \}$$

If $n = 1$, $m \in \mathbb{N}$ satisfies $m > 1$, then $m - 1 > 0$ & so $m-$ means $1 \in S$.

The Integers:

Def.: The set $\mathbb{Z} = \{ m \in \mathbb{R} \mid m \in \mathbb{N}$ or $m = 0$ or $-m \in \mathbb{N}$ is called the set of integers.

Propositⁱᵒⁿ 6: (i) $\mathbb{Z}$ is closed under additⁱᵒⁿ, substractⁱᵒⁿ & multiplicatⁱᵒⁿ

(ii) For any 2 integers k, l with $k > l$, we have $k - l \geq 1$

(iii) Every nonempty set $A \subseteq \mathbb{Z}$ that is bounded below has a minimum

(iv) — that is bounded above has a maximum.

Proof:

(i) additⁱᵒⁿ: let $m, n \in \mathbb{Z}$. Cases:

- $m, n \in \mathbb{N}$: nothing to prove
- m, n both 0: nothing to prove
- $-m, -n \in \mathbb{N}$: $m + n = -((-m) + (-n))$, which is in $\mathbb{Z}$ because $(-m) + (-n) \in \mathbb{N}$
- $m, -n \in \mathbb{N}$: • if $m = -n$: $m + n = 0 \in \mathbb{Z}$
  - if $m > -n$: $m + n = m - (-n) \in \mathbb{N} \subseteq \mathbb{Z}$
  - $m < -n$: $-(-m + n)$

⚠ if m in $\mathbb{N}$, then $-m$ in $\mathbb{Z}$

$3-2<3$ $\qquad x-a<r$

$x-2<5$

$-r+a<x<r+a$ $\qquad x+\ldots \quad x<5+2$

Similar for substract^e & multiplicat^e.

(iv) Let $A \subseteq \mathbb{Z}$ be nonempty & bounded below. Then, because $A \subseteq \mathbb{R}$, it has an infimum a.

(Recall: s supremum, $s-x<\varepsilon$)

there is an integer $m \in A$, with $m-a<1$.

m is the only integer in $[a, a+1)$.

Hence m is ~~ble~~ below all elmts of A that are not in $[a, a+1)$

Because m is the only elmt of A in $[a, a+1)$, m must be the minimum of A. ∎

Def. 5: • $\forall a \in \mathbb{R}\setminus\{0\}$, we set $\frac{1}{a} = a^{-1}$ & call it the reciprocal of a.

• For $b \in \mathbb{R}$ & $a \in \mathbb{R}\setminus\{0\}$ we set $\frac{b}{a} := b \cdot \frac{1}{a} = ba^{-1}$ & call it a fract^e.

Ex: $\frac{1}{2}+\frac{1}{2} = 2^{-1}+2^{-1} = (1+1)2^{-1} = 2 \cdot 2^{-1} = 1$

Theorem 5: (Archemedian Property of $\mathbb{R}$)

$\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ so that $n \geq x$

Proof: For a contradict^e, suppose x is s.t. $\forall n \in \mathbb{N}$ we have $n<x$.

Then $B = \{y \in \mathbb{R} \mid (\forall n \in \mathbb{N} \mid n<y)\}$ is not empty.

B is bounded below by all $n \in \mathbb{N}$.

Completeness Axiom $\Rightarrow$ B has an infimum, call it b.
Then $b - \frac{1}{2} \notin B$, which means there is an $n \in \mathbb{N}$ with $n \geq b$
But then $n+1 \geq b + \frac{1}{2}$ is a lower bound of B,
a contradict to $b = \inf(B)$.

The rational numbers:

Def. 6: • The set $\mathbb{Q} = \{\frac{n}{d} \mid n \in \mathbb{Z}, d \in \mathbb{Z}\}$ is called set of rational numbers
• $\mathbb{R} \setminus \mathbb{Q} := \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$ irrationals

prop 7: (i) $\mathbb{Q}$ closed under $+, -, \cdot$
(ii) If $q, r \in \mathbb{Q}$ & $r \neq 0$, then $\frac{q}{r} \in \mathbb{Q}$
(closed under divis.).

proof: (i) $\frac{m}{c} + \frac{n}{d} = mc^{-1} + nd^{-1} = m d d^{-1} c^{-1} + n c c^{-1} d^{-1}$
$= (md + nc)\, c^{-1} d^{-1}$
$= \frac{md+nc}{cd}$

Theorem 6: Let $a, b \in \mathbb{R}$ with $a < b$. Then there is a rational number $q \in \mathbb{Q}$ s.t. $a < q < b$

proof: By Theorem 5 (Archimedean Property of $\mathbb{R}$), there is an $n \in \mathbb{N}$ so that $0 < \frac{1}{b-a} < n$
By $\{0 < x < y \Rightarrow y^{-1} \leq x^{-1}\}$ we obtain $\frac{1}{n} < b - a$
Now let $u := \min\{m \in \mathbb{Z} \mid \frac{m}{n} \geq b\}$
$\cancel{l := \max\{m \in \mathbb{Z} \mid \frac{m}{n} \leq a\}}$

Then $\frac{u}{n} - \frac{\ell}{n} \geq b-a > \frac{1}{n}$ (9)

which means $\frac{\ell+1}{n} < \frac{u}{n}$

$\Rightarrow$ Hence, by def of $\ell+u$, we have $a < \frac{\ell+1}{n} < b$

Sect$^{\text{e}}$ 1.3: Inequalities & Identities

Def 1: For $x \in \mathbb{R}$, we set $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

absolute value of x

theorem 1: Properties

① $|x| \geq 0$ $\quad\quad \forall x \in \mathbb{R}$

② $|x| = 0$ iff $x = 0$

③ $\forall x, y \in \mathbb{R}$, $|xy| = |x| \cdot |y|$

④ Triangle inequality: $\forall x, y \in \mathbb{R}$, have $|x+y| \leq |x| + |y|$.

⑤ Reverse inequality: $||x| - |y|| \leq |x-y|$

⚠ **proof:** ②. "$\Leftarrow$" if $x = 0$, then by def. $|x| = |0| = 0$

. "$\Rightarrow$" let $x \in \mathbb{R}$ so that $|x| = 0$ & suppose for a contradict$^{\text{e}}$ that $x \neq 0$

If $x > 0$, then $0 < x = |x| = 0$, which is a contradic

Therefore, $x < 0$

But then $0 < -x = |x| = 0$, a contradict$^{\text{e}}$

Hence, the assumpt$^{\text{e}}$ was wrong, x must be equal to 0.

Def 2: For each $j \in \mathbb{N}$, let $a_j \in \mathbb{R}$. Define the sum $\sum_{j=1}^{1} a_j := a_1$ & for $n \in \mathbb{N}$ define sum

$$\sum_{j=1}^{n+1} a_j := a_{n+1} + \sum_{j=1}^{n} a_j$$

For $n \in \mathbb{N} \cup \{0\}$, set $\sum_{j=1}^{-n} a_j := 0$.

j summat$^{\underline{n}}$ index.

Def 3. product $\prod_{j=1}^{1} a_j := a_1$

$n \in \mathbb{N}$ $\prod_{j=1}^{n+1} a_j := a_{n+1} \cdot \prod_{j=1}^{n} a_j$

j product index

Def 5: . $\forall n \in \mathbb{N} \cup \{0\}$, define $n! := \prod_{j=1}^{n} j$ factorial

. $k \leq n$, binomial coeff $\binom{n}{k} : \frac{n!}{k!(n-k)!}$

Def 4. nth power $a^n := \prod_{j=1}^{n} a$

prop 1: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ $\quad k \leq n$

proof: $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!}$

⚠ 7.12, 19 [use 15]

Theorem: (Binomial Theorem) For all real numbers $a, b \in \mathbb{R}$ & all $n \in \mathbb{N}$, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

proof: Induct°, $P(n)$ statement about $(a+b)^n$

Base step: For $n = 1$

$$(a+b)^1 = a+b = \binom{1}{0} a^0 b^{1-0} + \binom{1}{1} a^1 b^{1-1}.$$
$$= \sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k}$$

Induct° step: holds for $n \rightarrow$ show for $n+1$

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$= \underbrace{\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k}}_{\text{let } j = k+1} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} a^j b^{n+1-j}}_{\text{let } j = k}$$

split off extra terms if needed.

use prop 1

HW → continue the proof

Sec$^{\circ}$ 2.1: The convergence of sequences

Def.: A sequence is a fct$^{=}$ $f: \mathbb{N} \to \mathbb{R}$
$n \mapsto f(n) = a_n$

a_n: n-th term of sequence
If the domain is $A \subset \mathbb{N}$ then we have a finite sequence.
e.g.: $\{1, \dots, 4\} \to \mathbb{R}$.

Start from n=0?

Ex 1: • $a_n = n$ (a_n) or $\{a_n\}_{n=1}^{\infty}$ (notat$^{\circ}$)
• $b_n = 2n$ (b_n)
• $c_n = 2^n$ (c_n)
• $d_n = 2^{-n} + 5n$ (d_n)

start w/ $2^0 = 1$ →
Pronoun
$\mathbb{N} \cup \{0\}$

"sequence = set w/ ordering."

Def. 2: A sequence (a_n) is decreasing if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$
increasing if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$
monotone neither

Def. 3: A nonempty subset A of a set X is called countable if it is in the range of some sequence
(not ⇒ uncountable)

Eg 2: The set of all rational #s in $[0, 1]$ is countable.
Sol^{n}: $\mathbb{Q} \cap [0,1] = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \dots\}$

in the range of sequence $n: 1, 2, 3, 4, 5, \dots$
$a_n: 0, 1, \frac{1}{2}, \frac{2}{3}, \frac{1}{4} \dots$

"stabilizes $\Leftrightarrow$ for a given tolerance ε

- there is a threshold N
- so that once the running index n has gone past the threshold N the sequence can only deviate from the limit by less than the tolerance ε.

Def 4: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real #s. Then $a \in \mathbb{R}$ is called limit of $\{a_n\}_{n=1}^{\infty}$ iff for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that for all $n \geq N$ we have $|a_n - a| <$

Quantifiers: $a \in \mathbb{R}$ is a limit of $\{a_n\}_{n=1}^{\infty}$ iff $\forall \varepsilon > 0$ $\exists N \in \mathbb{N} : \forall n \geq N : |a_n - a| < \varepsilon$

Prop 1: Let $x \in \mathbb{R}$. If $x \geq 0$ & for all $\varepsilon > 0$ we have $x \leq \varepsilon$, then $x = 0$

proof: Let x be as indicated & suppose for a contradict that $x > 0$. Then $\varepsilon := \frac{x}{2}$ is positive & $x \leq \varepsilon = \frac{x}{2}$ implies $1 \leq \frac{1}{2}$, a contradict $\blacksquare$

Theorem 1: Limits of sequences of real #s are uniq
i.e if $\{a_n\}_{n=1}^{\infty}$ have limits L & M, then $L = M$

proof: Let $\{a_n\}_{m=1}^{\infty}$ sequence & L & M its limits.

Need to prove $L = M$. By prop. 1, we do so by showing that for all $\varepsilon > 0$, the inequality $|L - M| < \varepsilon$ holds

- Let $\varepsilon > 0$ be arbitrary but fixed.

justify → Therefore there is an $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, we have $|a_n - L| < \frac{\varepsilon}{2}$

$N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$ $|a_n - M| < \frac{\varepsilon}{2}$

- Let $N = \max\{N_1, N_2\}$. Because $N \geq N_1$ for all $n \geq N$, we have $|a_n - L| < \varepsilon/2$

'' '' $|a_n - M| < \varepsilon/2$

$$|L - M| = |L - a_N + a_N - M| \quad \text{Then}$$
$$\leq |L - a_N| + |a_N - M| \quad \text{by } \Delta \text{ (triangle Inequality)}$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

By prop. 1, $|L - M| = 0$ ☒

Ex 3. prove that $\lim_{n\to\infty} \frac{3n-1}{2n+5} = \frac{3}{2}$

Sol: Let $\varepsilon > 0$ be arbitrary but fixed, & let $N \in \mathbb{N}$ be s.t. $N > \frac{17/\varepsilon - 10}{4}$. Then for all $n \geq N$, the following holds: $\left|\frac{3n-1}{2n+5} - \frac{3}{2}\right| = \left|\frac{6n-2}{4n+10} - \frac{6n+15}{4n+10}\right|$.

$$= \left|\frac{-17}{4n+10}\right|$$

$$= \frac{17}{4n+10} \quad n \in \mathbb{N} \text{ (any case, la}$$

$$< \frac{17}{4\cdot\frac{\frac{17}{\varepsilon}-10}{4}+10} = \frac{17}{17/\varepsilon} =$$

Theorem 2: (Limit Laws for Sequences) Let $\{a_n\}_{n=1}^{\infty}$ & $\{b_n\}_{n=1}^{\infty}$ be 2 convergent sequences. The followings hold.

① Sum $\{a_n+b_n\}_{n=1}^{\infty}$ converges & $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

②

③ Product $\{a_n \cdot b_n\}_{n=1}^{\infty}$ converges & $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty}$

④ If all $b_n \neq 0$ & $\lim_{n\to\infty} b_n \neq 0$, the quotient $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges & $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

Proof. (1) & (2) similar to proof Theorem 1

(3) Let $\varepsilon > 0$, Because $\lim_{n\to\infty} a_n = L$ there is an N_L so that for all $n \geq N_L$, we have (i) $\boxed{|a_n - L| < \frac{\varepsilon}{2}\alpha}$

to be dete

- Similarly, cuz $\lim_{n\to\infty} b_n = M$, there is $N_M \in \mathbb{N}$, s.t. $n \geq N_M$ (ii) $\boxed{|b_n - M| < \frac{\varepsilon}{2}\cdot\beta}$?

- Finally, becuz $\lim_{n\to\infty} a_n = L$, there is $K_1 \in \mathbb{N}$ s.t. for all $n \geq K_1$, we have $|a_n - L| < 1$.

Consequently, $|a_n| - |L| \leq \big||a_n| - |L|\big|$
$\leq |a_n - L| < 1$ (reverse Δ).

Thus, (iii) $\boxed{|a_n| < |L| + 1.}$

- Let $\boxed{N = \max\{N_L, N_M, K_1\}}$, then for all $n \geq N$, the inequalities hold: (i), (ii), (iii)

- For all $n \geq N$, we obtain $|a_nb_n - LM| = |a_nb_n - a_nM + a_nM - LM|$

$\leq |a_nb_n - a_nM| + |a_nM - LM|$ by Δ

$= |a_n||b_n - M| + |a_n - L||M|$

$< (|L| + 1)|b_n - M| + |a_n - L||M|$

$< \boxed{(|L|+1)\beta\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\cdot\alpha|M|}$
(1) (2)

from (i)   from (ii)

① ↓   ② ↓

& $\beta = \dfrac{1}{(|L|+1)}$ $\alpha = \dfrac{1}{|M|}$ need to be ①?

Problem: $|M|$ maybe 0

soluᵗ: $\alpha = \dfrac{1}{(|M|+1)}$

/17/09

Def 1. A sequence $\{a_n\}_{n=1}^{\infty}$ is called bounded above iff $\exists$ $A \in \mathbb{R}$ (upper bound)

s.t. for all $n \in \mathbb{N}$, the inequality $a_n \leq A$ holds.

bounded below iff $\exists$ $B \in \mathbb{R}$ .. $a_n \geq B$ (lower bound)

- Sequence is bounded iff it is bounded above and bounded below otherwise, unbounded.

Exple: $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ bounded (between 0 & 1)

$\{n\}_{n=1}^{\infty}$ bounded below, so unbounded

Theorem 1: Any convergent sequence of real numbers is bounded.

proof: Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence & let $L = \lim_{n \to \infty} a_n$

- Need to prove: $\exists$ $M \in \mathbb{R}$ s.t. for all $n \in \mathbb{N}$ the inequality $|a_n| \leq M$ hold ($|a_n| \leq M$)

- Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ so that for all $n \geq 1$ we have $|a_n - L| < \varepsilon$.

[Cheat] Now let $M := \max\{|L| + \varepsilon, |a_1|, \dots, |a_{N-1}|\}$.
Then for all $n < N$, we have $|a_n| \leq M$
& for $n \geq N$ we obtain $|a_n| \leq |a_n - L| + |L|$ (Δ

$$< \varepsilon + |L| \leq M$$

Have proved that $\{a_n\}_{n=1}^{\infty}$ is bounded below by $-M$ & above by M ▨

Ex 2: The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but it does not converge.

Def 2: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Then $\{a_n\}_{n=1}^{\infty}$ is called nondecreasing (monotonically increasing) iff for all $n \in \mathbb{N}$ we have $a_n \leq a_{n+1}$

Nonincreasing (monotonically decreasing) iff ...... $a_n \geq a_{n+1}$

- If $\{a_n\}$ is either nondecreasing or nonincreasing, called monotone.
- $\{a_n\}$ is (strictly) increasing iff $\forall n \in \mathbb{N}$ we have $a_n < a_{n+1}$
- $\{a_n\}$ is (strictly) decreasing $a_n > a_{n+1}$
- $\{n\}_{n=1}^{\infty}$ shows that nondecreasing sequences can grow beyond all bounds. But if this is not the case, a monotone sequence converges.