MATH 404

by Yong Zheng

- linear programming → ch 9
- finite-state Markov chain → ch 10

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2 midterms	200 pts	} Grader
5 HWs	100 pts	} Grader
10 Quizzes	200 pts	→ Myself
1 Final	200 pts	→ teacher

Review:

Solve a system of Linear System of Equat^{k}

Gauss eliminat$^{\circ}$ method

Definit$^{\circ}$: (linear equat$^{\circ}$) A linear eqt^{k} in $x_1, x_2, \cdots x_n$ is eqt$^{\circ}$ in the following form : $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.

(linear cuz exponent of x is 1)

- A system of linear Eqt^{k} is a collect^{k} of LES in the sa variables.

Exple:

$$\begin{cases} 2x_1 - x_2 + \frac{3}{2}x_3 = 8 \\ x_1 \quad - 4x_3 = -7 \end{cases} \qquad (2)$$

- A solut of a system of LES is a list of numbers $(s_1, s_2, \cdots, s_n)$ that makes each eqt$^{\circ}$ a true statem when the values $s_1, s_2, \cdots, s_n$ are substituted for x_1, x respectively.

Exple $(5, \frac{13}{2}, 3)$ is a solut$^{\circ}$ of (2)

Matrix notat$^{\circ}$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & & & \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (3)$$

augmented matrix. $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$

coefficient matrix $= A$ $\qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

<u>Theorem</u>: system (3) has either
- no solut$^{\underline{s}}$
- exactly one solut$^{\underline{o}}$
- infinitely many solut$^{\underline{s}}$

<u>Definit$^{\underline{o}}$</u>: 2 systems are <u>equivalent</u> if they both have the <u>same</u> <u>solut$^{\underline{s}}$</u>.

e.g. $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \Leftrightarrow \begin{cases} (a_{11} + 4a_{21})x_1 + (a_{12} + 4a_{22})x_2 = b_1 + 4b_2 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

⚠ 3 elementary row operat^{ns}:

- replacement
- interchange
- scaling

Exple.
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

solut^{e}

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 3 \\ -4 & 5 & 9 & -9 \end{bmatrix} \xrightarrow[\text{by [row 3]} + 4\times\text{[row 1]}]{\text{replace [row 3]}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$\downarrow$ multiply [row 2] by $\frac{1}{2}$

$$\begin{array}{r} x_1 - 2x_2 + x_3 = 0 \leftarrow \\ x_2 - 4x_3 = 4 \leftarrow \\ x_3 = 3 \leftarrow \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xleftarrow[\text{by [row 3]} + 3\text{[row 2]}]{\text{replace [row 3]}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\Downarrow$$

$$x = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$$

Theorem: A system $Ax = b$ has a solut^{e}
$\iff$ the echelon form of the augmented matrix does NOT have a row like $[0, 0, 0, \cdots, b]$ $b \neq 0$

Exple: choose h & k s.t. the system

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

has: • no solut^{e}
• a uniq solut^{e}
• many solut^{es}

solut^{n}:
$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

• if $8-4h=0$ & $k-8 \neq 0$, i.e $h=2$ $k \neq 8$
then the system doesn't have a solut^{e}
• if $8-4h \neq 0$, i.e. $h \neq 2$,
then the system has a uniq solut^{e}
• if $8-4h=0$ & $k-8=0$, i.e. $h=2$ & $k=8$,
then the system has infinitely many solut^{s}.

Cramer's rule: $Ax = b$
1. # of unknown variables = # of eq^{ts}
(coefficient matrix is a square matrix)
2. $\det A \neq 0$ $\quad x_i = \dfrac{\det A_i}{\det A}$ | A_i = matrix formed when we replace the ith column by b

Exple

$$\begin{cases} 2x_1 + x_2 = 5 \\ x_1 + x_2 = 3 \end{cases} \qquad \begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \det A = 1$$

$$A_1 = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} \Rightarrow \det A_1 = 2$$

$$A_2 = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \Rightarrow \det A_2 = 1$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{2}{1} = 2$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{1}{1} = 1.$$

HW sect$^{\circ}$ 1.1: 14.16

s sect$^{\circ}$ 1.2: 20

Sect° 1.5: Homogeneous linear systems.

Def.: A system of linear eq^{es} is homogeneous if it can be written as $A\,\underline{x} = \underline{0}$

$$A: m \times n$$

$$\underline{x} = n \times 1$$

$$\underline{0} = m \times 1$$

$Ax = 0$ must have at least 1 solut^e $x = \bar{0}$

↳ trivial solut^e.

We are going to try to find a non-trivial solut^e to $Ax = 0$

↳ not all $x_1, \dots, x_n$ are 0.

Theorem: $Ax = 0$ has a non-trivial solut^e iff it has at least 1 free variable

e.g.: $\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$

$$Ax = 0, \quad A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} 3x_1 - 4x_3 = 0 \\ x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$\begin{cases} Ax = b \\ Ay = 0 \end{cases}$$

$\Rightarrow A(x+y) = b$

$Ax = b$

$\cancel{x}y = \underbrace{x_H}_{\text{Homogeneous solut}^{n}} + \overbrace{p}^{\text{specific solut}^{e}}$

e.g. $Ax = b$

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \qquad b = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

$$x_H = \begin{pmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$[A \mid b] = \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$3x_1 - 4x_3 = -3$

$x_2 = 2$

$$\begin{cases} x_1 = \frac{4}{3} x_3 - 1 \\ x_2 = 2 \end{cases} \Rightarrow p = \begin{pmatrix} 4/3\, x_3 - 1 \\ 2 \\ x_3 \end{pmatrix}$$

set $x_3 = 0, 1, 2, 3, \ldots$

for $x_3 = 1 \rightarrow p = \begin{pmatrix} 1/3 \\ \cdots \end{pmatrix}$

$$\begin{matrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -5 & h+4 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & h-6 \end{pmatrix}$$

$h = 6$

So the general solut° here of $Ax = b$ is:

$$x_g = \underbrace{x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}}_{x_H \text{ (don't set } x_3 \text{ here)}} + \underbrace{\begin{pmatrix} 1/3 \\ 2 \\ 1 \end{pmatrix}}_{\text{set fix } x_3}$$

Theorem 6: suppose $Ax = b$ is consistent for some b & let p be a solut° $\Rightarrow x_g = x_H + p$

Sect° 1.7:

- Linearly independent:

  Vectors $S = \{ \underline{v_1}, \underline{v_2}, \dots \underline{v_p} \}$ in $\mathbb{R}^n$ $\quad \left(V_i = \begin{pmatrix} x \\ x \\ \vdots \end{pmatrix}_n \right.$

  are linearly independent if

  $x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_p \underline{v_p} = 0$ has only the trivial solut°

  as in $x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \bar{0}$ (doesn't have the non-trivial sol

- Linearly dependent:

  if there exist weights $c_1, c_2, \dots, c_p$ not all 0

  s.t. $c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_p \underline{v_p} = 0$

  (has non-trivial solut°

$$x_1 \bar{v_1} + x_2 \bar{v_2} + \dots + x_p \bar{v_p} = 0 \Leftrightarrow (\bar{v_1}, \bar{v_2}, \dots, \bar{v_p}) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = 0$$

$$\Leftrightarrow A_{n \times p} \cdot x = 0$$

if $Ax = 0$ $\begin{cases} \text{has non-trivial solut° } \Rightarrow \text{ dependent} \to \text{all columns vectors in } A \\ \text{doesn't have a non-trivial solut° } \Rightarrow \text{ independent} \end{cases}$

$c = \frac{m}{n}$

$a \nmid$

$n \cdot b$

linearly dependent

$\Uparrow$

free variable(s) $\Rightarrow$ Non-trivial solut^{e} e.g. $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

NB

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e.g. $A = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 0 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{pmatrix}$

No free variable $\Rightarrow$ independent

$c = \frac{m}{n}$, $a \nmid q$, $b \nmid q$, $ab \nmid q$, $ab \nmid q$

theorem 8: if a set contains more vectors than the # of entries in each vector $\Rightarrow$ set is linearly dependent.

$S = \{v_1, v_2, \dots v_p\}$ in $\mathbb{R}^n$

if $p > n \Rightarrow S$ are dependent.

Theorem 9: if $S = \{\bar{v_1}, \bar{v_2} \dots \bar{v_p}\}$ in $\mathbb{R}^n$ contains a $\bar{0}$ vector $\Rightarrow$ set are linearly dependent

proof: $v_1, v_2, \dots v_p$

if $v_1 = 0$, $1 \cdot 0 + 0 \cdot v_2 + 0 \cdot v_3 + \dots + 0 \cdot v_p = 0$

$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

e.g. $\begin{pmatrix} -4 & -3 & 0 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \\ 0 & -1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & -3 & 12 \\ 0 & -1 & 4 \\ 0 & +1 & 24 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 28 \\ 0 & 0 & 84 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 28 \\ 0 & 0 & 84 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

trivial solut^{e} $\Rightarrow$ linearly independent

19/08/09 HW due sep 17 together w/ 1rst HW:

sec 1.5: 3, 5

sec 1.7: 5

sec 1.8: 4

sec 1.9: 15

sec 2.2: 2,3

Sect° 1.8:

Def: transformat° T

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

domain of T — range of T

e.g.: $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$ $a = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$b = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$

$c = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

by $T(\bar{x}) = A\bar{x}$

(a) find $T(u)$.

$T(u) = Au = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$

(b) find an $\bar{x}$ in $\mathbb{R}^2$ whose image under T is $\bar{b}$.

find $\bar{x}$ s.t. $T(\bar{x}) = \bar{b} = A\bar{x}$

so $\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right)$

so $\bar{x} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$

(c) is there more than one $\bar{x}$ whose image under T is $\bar{b}$?

No. the solut$^{\circ}$ is uniq according to (b).

(d) Determine if $\bar{c}$ is the range of **T**.

see if there is a solut $\bar{x}$ for $T(\bar{x}) = \bar{c}$

$$A\bar{x} = \bar{c}$$

$$\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$[A \mid c] = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2^{1/2} \end{pmatrix} \rightarrow 0x_1 + 0x_2 = 2^{1/2}$$

$$0 = 2^{1/2})$$

So there is no solut

Therefore $\bar{c}$ is not in the range of T.

<u>Linear Transformat^{e}</u>:

$$T: \text{ if } \begin{cases} (1)\ T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}) & , \bar{u} \,\&\, \bar{v} \text{ in Domain of } \\ (2)\ T(c.\bar{u}) = c.T(\bar{u}) & , c \text{ is a scalar.} \end{cases}$$

$\Rightarrow$ T is linear

e.g: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^1$

$$T(\bar{x}) = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \bar{x}_1$$

is T linear?

select $\bar{u}, \bar{v}$ in $\mathbb{R}^2$

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$L.H.S. = T(\bar{u} + \bar{v}) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = u_1 + v_1$$

$$R.H.S. = T(\bar{u}) + T(\bar{v}) = T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \bar{u}_1 + \bar{v}_1$$

L.H.S = Left Hand Side

(2) $T(c.\bar{u}) = T\left(c\begin{pmatrix}u_1\\u_2\end{pmatrix}\right) = T\begin{pmatrix}cu_1\\cu_2\end{pmatrix} = cu_1$

R.H.S = $cT(\bar{u}) = c.T\begin{pmatrix}u_1\\u_2\end{pmatrix} = c.u_1$

If T is linear,

then (1) $T(\bar{0}) = \bar{0}$

(2) $T(c.\bar{u} + d.\bar{v}) = cT(\bar{u}) + dT(\bar{v})$

(3) $T(c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_p\bar{u}_p) = c_1T(\bar{u}_1) + c_2T(\bar{u}$
$+ c_pT(\bar{u}_p$

$c_1, \ldots c_p$ scalars

proof:

(1) $T(\bar{0}) = T(\bar{0} + \bar{0})$
$= T(\bar{0}) + T(\bar{0}) \Rightarrow T(\bar{0}) = \bar{0}$

(2) $T(c\bar{u} + d\bar{v}) = T(c\bar{u}) + T(d\bar{v})$
$= cT(\bar{u}) + dT(\bar{v})$

e.g.: $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\bar{x}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-x_2\\x_1\end{pmatrix}$

find the image under T of $\bar{u} = \begin{pmatrix}4\\1\end{pmatrix}$, $\bar{v} = \begin{pmatrix}2\\3\end{pmatrix}$, $\bar{u}+\bar{v} = ($

Ans. $T(\bar{u}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$

$T(\bar{v}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}$

$T(\bar{u}+\bar{v}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}-4\\6\end{pmatrix}$

3×2 · 2×1 → 3×1

Sec 1.9:

e.g. $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

suppose T is linear transform from $\mathbb{R}^2$ to $\mathbb{R}^3$

s.t. $T(e_1) = \begin{pmatrix}5\\-7\\2\end{pmatrix}$ $T(e_2) = \begin{pmatrix}-3\\8\\0\end{pmatrix}$

find a formula for the image of an arbitrary $\bar{x}$ in $\mathbb{R}^2$

set $\bar{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$

Ans: $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\0\end{pmatrix} + \begin{pmatrix}0\\x_2\end{pmatrix}$

$= x_1 e_1 + x_2 e_2$

$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = T(x_1 e_1 + x_2 e_2)$

$= x_1 T(e_1) + x_2 T(e_2)$

$T\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1\begin{pmatrix}5\\-7\\2\end{pmatrix} + x_2\begin{pmatrix}-3\\8\\0\end{pmatrix}$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformat

Then there exist a uniq A s.t. $T(\bar{x}) = A\bar{x}$ for all $\bar{x}$ in $\mathbb{R}^n$.

e.g.: find all missing #s

$$\begin{bmatrix}1 & -1\\ -2 & 1\\ 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 - x_2\\ -2x_1 + x_2\\ x_1\end{pmatrix}$$

Sec 2.2: Inverse Matrix

An $n\times n$ matrix is invertible there exist $n\times n$ C

s.t. $CA = I$

$AC = I$

$\Rightarrow C = A^{-1}$

$\boxed{AA^{-1} = A^{-1}A = I}$

Theorem 4: $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

(i) if $ad - bc \neq 0$ (det A), then A is invertible $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

(ii) if $ad - bc = 0$

$\Rightarrow A$ is not invertible

Theorem 5: If A is invertible n then for any $\bar{b}$ in $\mathbb{R}^n \Rightarrow A\bar{x} =$ have a uniq solut $\bar{x} = A^{-1}$.

Property: A, B invertible

(i) A^{-1} invertible / $(A^{-1})^{-1} =$

(ii) AB is invertible

$(AB)^{-1} = B^{-1}A^{-1}$

(check: $(AB)\cdot B^{-1}A^{-1} = A(B\cdot B^{-1})A^{-1} = AA^{-}$

$$-40-(-35) = -40+35 = -5$$

iii) A^T is invertible $\Rightarrow (A^T)^{-1} = (A^{-1})^T$

(check: $A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I^T = I$

⚠ An Algorithm A^{-1}

$$[A \mid I] \xrightarrow[\text{operates}]{\text{row}} [I \mid A^{-1}]$$

e.g. $A = \begin{pmatrix} 5 & 10 \\ 4 & 7 \end{pmatrix}$

NBA

$$[A \mid I] \leftrightarrow \left(\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4 & -1 \end{array}\right)$$

$\frac{1}{5} - \frac{40}{5}$

$35 - 40 = -5$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -39/5 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right)$$

$-\frac{1}{5}\begin{pmatrix} 7 & -10 \\ -4 & 5 \end{pmatrix}$

$$\cancel{A^{-1} = \begin{bmatrix} -39/5 & 2 \\ 4 & -1 \end{bmatrix}}$$

$$A^{-1} = \begin{pmatrix} -7/5 & 2 \\ 4/5 & -1 \end{pmatrix}$$

Linear Programming Problem (LPP) — Geometric method

Example 1: A car manufacturer makes 2 cars

	Assembly	Finishing	Profit
Model A	4 hours	6 hours	400
Model B	6 hours	3 hours	300

Assume it has 720 hours of assembly time & 480 hours of finishing time available

Q: How many of each Model will it make s.t. it maximizes its profit

Solut: Let x_1 be the number of unit of Model A to be made

x_2 . . . of Model B . . .

- Total Profit: $400x_1 + 300x_2$ (we need to maximize this ↑)
- constraints

$$4x_1 + 6x_2 \le 720$$
$$6x_1 + 3x_2 \le 480$$
$$x_1 \ge 0, \; x_2 \ge 0$$

objective fct^e

Maximize $400x_1 + 300x_2$ subject to $4x_1 + 6x_2 \leq 720$

$6x_1 + 3x_2 \leq 480$

& $x_1 \geq 0 , x_2 \geq 0$

Canonical form of LPP:

Find $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} \in \mathbb{R}^n$ to MAXIMIZE $f(\bar{x}) = c_1x_1 + c_2x_2 + \dots + c_nx$

with the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_n$$

& $x_j \geq 0 \quad j = 1, 2, \dots n$

Matrix notat^e:

$C = \begin{bmatrix} c_1 \\ c_2 \\ c_n \end{bmatrix} \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$

Maximize $f(\bar{x}) = C^T\bar{x}$ $\longrightarrow$ objective fct^e

subject to $A\bar{x} \leq b$ & $\bar{x} \geq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

HW: sect^{n} 9.2
2, 4, 7, 8
Due sep 17

Definit^{n}: (feasible solut^{n} set) $\mathcal{F}$

$$\mathcal{F} = \{x \in \mathbb{R}^n : Ax \le b \;\&\; \bar{x} \ge 0\}$$

Exple: what's the feasible solut^{n} set of LPP in exple 1?

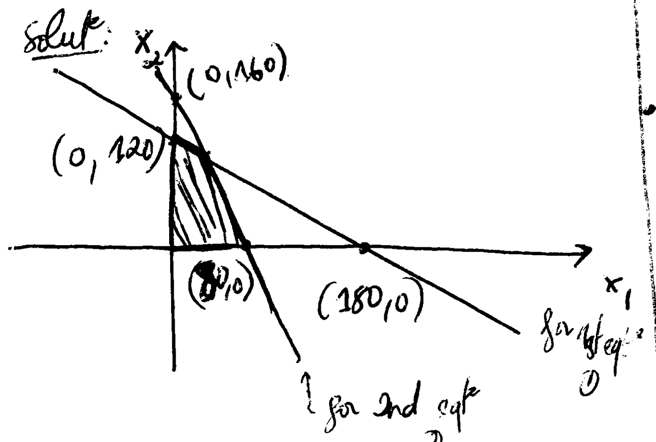
Maximize $400x_1 + 300x_2$

subject to $4x_1 + 6x_2 \le 720$ ①

$6x_1 + 3x_2 \le 480$ ②

& $x_1 \ge 0$ & $x_2 \ge 0$.

Solut^{n}:



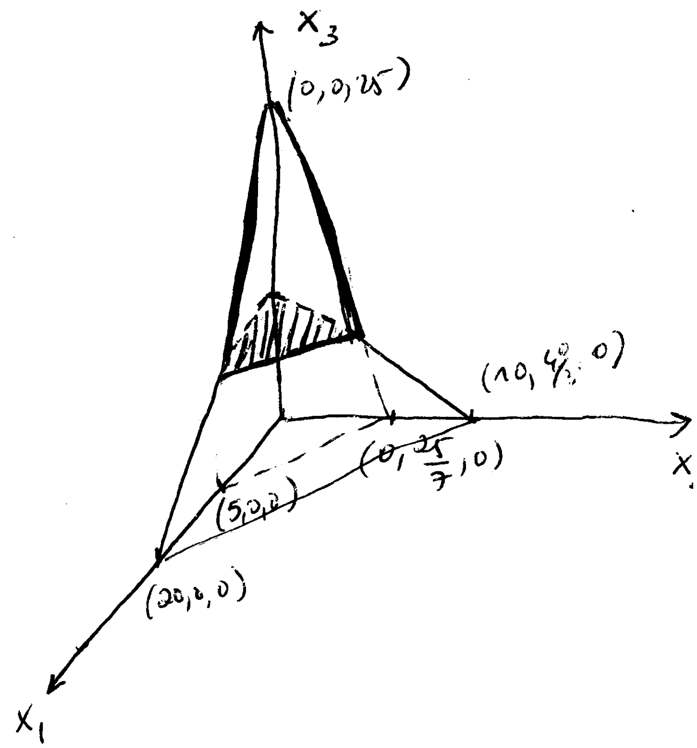
Exple: Find the feasible solut^{n} set of LPP

minimize $3x_1 + x_2 + 5x_3$

subject to $5x_1 + 7x_2 + x_3 \le 25$

$2x_1 + 3x_2 + 4x_3 = 40$

& $x_1 \ge 0$, $x_2 \ge 0$.



- Convert minimize $f(\bar{x}) = C^T x$ into maximize $-f(\bar{x}) = -C^T\bar{x}$
- Convert $d^T x \ge S$ into $-d^T x \le -S$
- convert $d^T x = S$ into $d^T x \le S$, $-d^T x \le -S$

Theorem: For a given canonical LPP, if $\mathcal{F} \ne \phi$ & objective fct^{n} is bounded on $\mathcal{F}$, then LPP has at least one optimal solut^{n}.

Futhermore, at least one optimal solut^{n},

1) - Write LPP into canonical form
2) -
3) - picture
4) - find the max value of objective ~~f~~ & the ~~opti~~ corresponding pt is the optimum

the vertex of $\mathcal{F}$

Example: solve the LPP in exple 1.

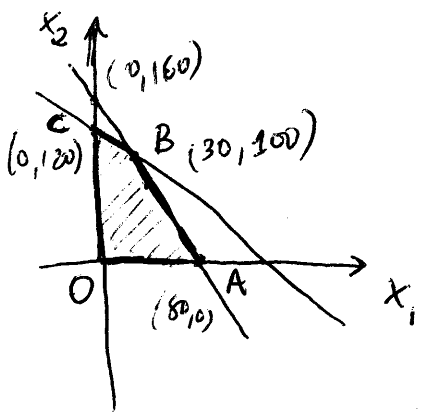
Maximize $400x_1 + 300x_2$

subject to $4x_1 + 6x_2 \leq 720$

$6x_1 + 3x_2 \leq 480$

& $x_1 \geq 0$, $x_2 \geq 0$

solut^{e}:



$\mathcal{F} \neq \phi$

the objective fct^{e} is bounded $\Rightarrow$ it has an optimal solut^{e}.

$$400x_1 + 300x_2 = \begin{cases} 0 & \text{if } (x_1, x_2) = (0,0) \\ 32\,000 & \text{if } (x_1, x_2) = (80,0) \\ 42\,000 & \text{if } (x_1, x_2) = (30,100). \\ 36\,000 & \text{if } (x_1, x_2) = (0,120) \end{cases}$$

Thus $(x_1, x_2) = (30, 100)$ is one optimal solut^{e}

Exple: Minimize $3x + 2y$

subject to $3x + y \geq 5$

$x + y \geq 3$

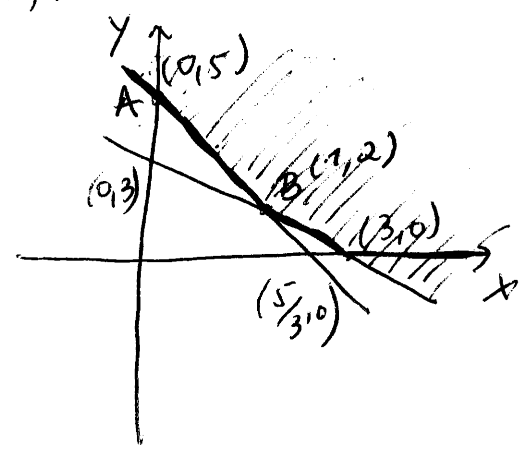
& $x \geq 0$, $y \geq 0$

solut^{e}:

1) Convert LPP into Canonical fo

$$\boxed{\begin{array}{ll} \text{Maximize} & -3x - 2y \\ \text{subject to} & -3x - y \leq -5 \\ & -x - y \leq -3 \\ & \& \ x \geq 0,\ y \geq 0. \end{array}}$$

2) Picture



vertex	value of objective fct^{e}
(0,5)	-10
(1,2)	-7
(3,0)	-9

optimal solu

HW2 sect^e 9.3
9, 10

Due ~~sept 17~~

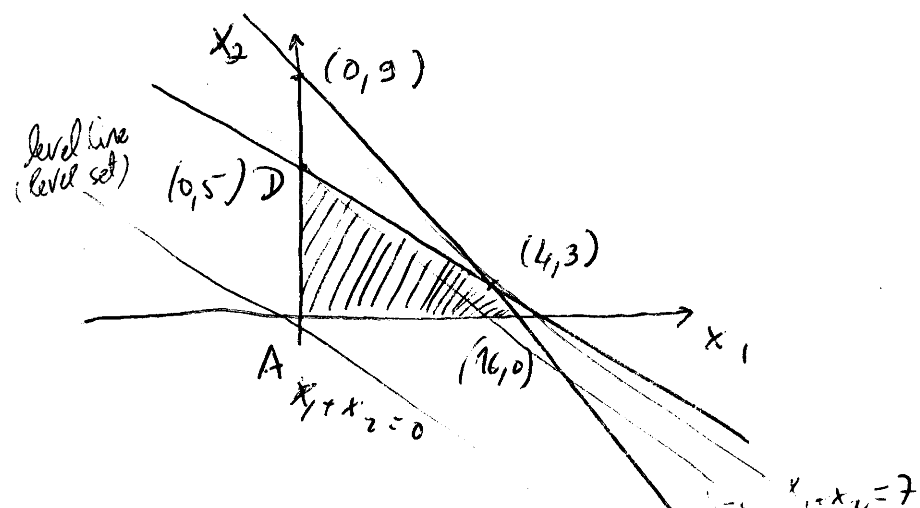
Geometric method:

Expl: Maximize $x_1 + x_2$

subject to $x_1 + 2x_2 \leq 10$

$3x_1 + 2x_2 \leq 18$

$x_1 \geq 0, \quad x_2 \geq 0$



Solut^e: the feasible solut^e set is the shaded area

Coordinates	value of OF (Objective Fct^e) ($x_1 +$
$A = (0,0)$	0
$B = (6,0)$	6
$C = (4,3)$	7 ←
$D = (0,5)$	5

⚠ (the corresponding lines are parallel.)

Thus $(4,3)$ is an optimal solut^e.

⚠ **Slack variables:** (sect° 9.3)

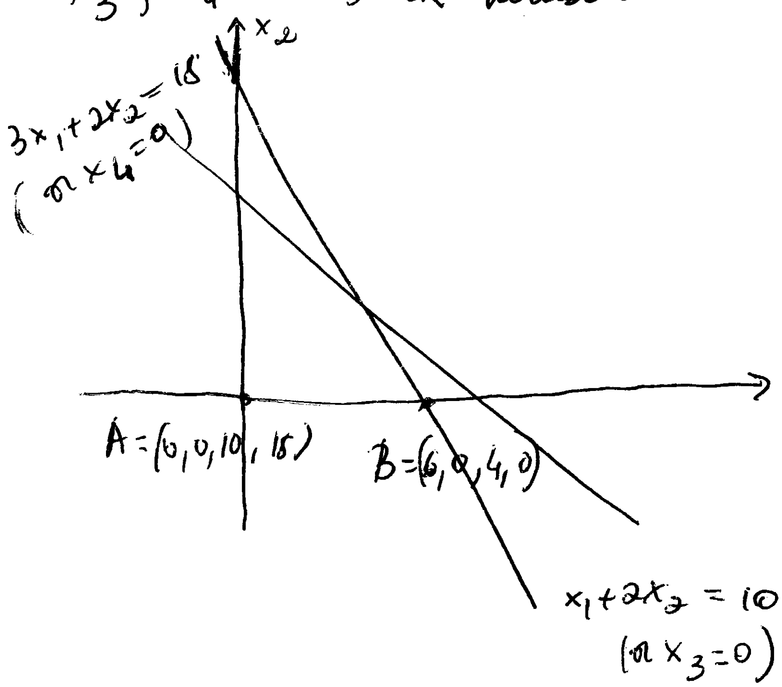
Add new variables to constraints & convert them to a system Linear Equal° (LE):

$$x_1 + 2x_2 + x_3 = 10$$

$$3x_1 + 2x_2 + x_4 = 18$$

$$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0$$

x_3, x_4 are slack variables



$$x_3 \geq 0 \Rightarrow x_1 + 2x_2 \leq 10$$

$$x_4 \geq 0 \Rightarrow 3x_1 + 2x_2 \leq 18$$

Feasible solut° set: $\{(x_1, x_2, x_3, x_4); \; x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0\}$

e.g.: Solve LPP:

maximize $u = 4x + 3y$

subject to $x + y \leq 4$

$-x + y \leq 2$

& $x \geq 0$, $y \geq 0$

Solut: Add slack variables to constraints & convert them to a system of LE.

$x + y + s = 4$

$x + y + r = 2$

$x \geq 0$, $y \geq 0$, $s \geq 0$, $r \geq 0$

pts		$u =$
$(0, 0, 4, 2)$	: vertex of feasible solut set	0
$(0, 4, 0, -2)$	(not a vertex)	(not a vertex)
$(0, 2, 2, 0)$		6
$\boxed{(4, 0, 0, 6)}$		$\boxed{16}$ ← optimal pt is $(4, 0,$
$(-2, 0, 6, 0)$		(not a vertex)
$(1, 3, 0, 0)$		13

Some definit^{ns}:

Maximize $f(x) = C^T X$

s.t. $Ax \leq b$

& $x \geq 0$

where $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

A $m \times n$ matrix $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

- standard slack form of LPP:

Exple: Max $u = 3x + 4y$

s.t. $x + 2y \leq 12$

$2x - 3y \leq 18$

& $x, y \geq 0$

Add slack variables it becomes

Max $u = 3x + 4y$

s.t. $x + 2y + s = 12$

$2x - 3y + r = 18$

& $x, y, r, s \geq 0$

write this system in the following form:

Max $u = 3x + 4y$

s.t. $x + 2y - 12 = -s$

$2x - 3y - 18 = -r$

& $x, y, s, r \geq 0$

$\rightarrow$ this is a standard slack form.

⚠ s, r are basic variable
x, y are nonbasic variables

- Canonical form of LPP for simplex method:
  1) the objective fct is to be maximized
  2) In each equat^e, there is a variable only contained in the eq^{te} & its coefficient is 1.

- Basic variables & nonbasic variables:
  Assume nonbasic variables equal to 0 & solve the basic variables
  Then we have a solut^e of constraint (system of LEs)
  we call this solut^e "<u>basic point</u>."
  If all entries of a basic point are <u>nonnegative</u>, then we call it <u>basic feasible solut^e</u>.

<u>Theorem:</u>
If LPP has optimal solut^e, then one of the optimal solut^e is at least one <u>basic feasible solut^e</u>.

5. Perfect canonical form of LPP for simplex method:
- LPP is in canonical form
- the constant term ≥ 0
- the objective fct^e is expressed in terms of NONBASIC variables.

Exple: Maximize $U = 2x + y$

s.t. $x + 2y + s = 12$

$2x - y + 3r + t = \boxed{-19}$

& $x, y, r, s, t \geq 0$

[This ~~is not~~ in perfect canonical form cuz the a constant term < 0 (-19).]

(* we try to convert it into a perfect canonical form*)

* pivot operat°

Example: Maximize $u = 4x - 3y + 2z$

s.t. $2x + y + z \leq 12$

$-x + 2y + 3z \leq 9$

$3x + 4y - z \leq 6$

& $x, y, z \geq 0$

Add slack variables. this becomes

Max $u = 4x - 3y + 2z$

s.t. $2x + y + z + s = 12$

$-x + 2y + 3z + r = 9$

$3x + 4y - z + t = 6$

& $x, y, z, s, r, t \geq 0$

This LPP is in perfect canonical form

write LPP in standard slack form.

Max $u = 4x - 3y + 2z$

s.t $2x + y + z - 12 = -s$

$-x + 2y + 3z - 9 = -r$

$3x + 4x - z - 6 = -t$

& $x, y, z, s, r, t \geq 0$

$\frac{(1\times3)}{2}$

$-1 -$

$5/2 = 2 - \frac{1(x-1)}{2}$ $\quad -5/2 = -1 - \frac{(1\times3)}{2}$

pivot elmt

	x	y	z	1	
eqt	2	1	1	-12	= -s
1 eqt	-1	2	3	-9	= -r
2 eqt	3	4	-1	-6	= -t
the goal	4	-3	2	0	= u

s	y	z	1	
1/2	1/2	1/2	-6	= -x
1/2	5/2	7/2	-15	= -r
-3/2	5/2	-5/2	12	= -t
-2	-5	0	24	= u

- replace pivot elmt by its reciprocal
- divide every other entry in the pivot row by the pivot elmt.
- divide every other entry in the pivot column by (- pivot elmt)
- ...
- Interchange the posit^e of x & s.

1/22/09 [HW online]

Example 1: solve Max $u = 4x - 3y + 2z$

s.t. $2x + y + z \leq 12$

$-x + 2y + 3z \leq 9$

$3x + 4y - z \leq 6$

& $x, y, z \geq 0$

solut^e: ① Add slack variables to constraints

Max $u = 4x - 3y + 2z$

s.t $2x + y + z + r = 12$

$-x + 2y + 3z + s = 9$

$3x + 4y - z + t = 6$

& $x, y, z, r, s, t \geq 0$.

LPP is in perfect canonical form

② write LPP in standard slack form:

Max $u = 4x - 3y + 2z$

s.t. $2x + y + z - 12 = -r$

$-x + 2y + 3z - 9 = -s$

$3x + 4y - z - 6 = -t$

w/ $x, y, z, r, s, t \geq 0$

③ Establish the simplex tableau

x	y	z	1	
2	1	1	-12	$= -r$
-1	2	3	-9	$= -s$
3	4	-1	-6	$= -t$
4	-3	2	0	$= \mu$

From this tableau, we assume $x = y = z = 0$

then $r = 12$, $s = 9$, $t = 6$, $\mu = 4x - 3y + 2z + 0 =$

this basic feasible solut° is NOT optimal.

pivot operat°

make x basic (since the coef of x is the largest positiv number.

$a_{11} = 2 > 0$

$a_{31} = 3 > 0$

$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 6 \end{bmatrix}$ $\frac{b_1}{a_{1r}} = \frac{12}{2} = 6$ $\frac{b_3}{a_{31}} = \frac{6}{3} =$ [illegible]

⚠ the variable corresponding to the smallest ratio is to be nonbasic.

pivot the circled 3 & the tableau becomes the following

	t	y	z	1	
	$-2/3$	$-5/3$	$5/3$	-8	$= -r$
	$1/3$	$19/3$	$8/3$	-11	$= -s$
	$1/3$	$4/3$	$-1/3$	-2	$= -x$
	$-4/3$	$-25/3$	$10/3$	8	$= u$

$r = 8 \quad s = 11 \quad x = 2$

$u = 8 = -\frac{4}{3}t - \frac{25}{3}y + \frac{10}{3}z$

it's not optimal too

Apply pivot operat$^{\underline{e}}$ one more time

make z basic

$a_{13} = \frac{5}{3}$

$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 2 \end{bmatrix}$

$a_{23} = \frac{8}{3}$

$\frac{b_1}{a_{13}} = \frac{8}{5/3} = \frac{24}{5}$

$\frac{b_2}{a_{23}} = \frac{11}{8/3} = \frac{33}{8}$

$\frac{b_1}{a_{13}} =$

pivot the circled $\frac{8}{3}$ the tableau becomes:

	t	y	s	1	
	$-7/8$	$-15/4$	$-5/8$	$-9/8$	$= -r$
	$1/8$	$5/4$	$3/8$	$-33/8$	$= -z$
	$3/8$	$7/4$	$1/8$	$-27/8$	$= -x$
	$-7/4$	$-25/2$	$-5/4$	$87/4$	$= u$

$t = y = s = 0 \qquad r = 9/8, \; z = 33/8, \; x = \frac{27}{8}$

$u = 87/4 \qquad$ i.e. $(x, y, z) = (27/8, 0, \frac{23}{8})$ is one optimal solution this is equivalent to doing:

Max $u = -\frac{4}{3}t - \frac{25}{3}y + \frac{10}{3}z + 8$

s.t $\frac{-2}{3}t - \frac{5}{3}y + \frac{5}{3}z - 8 = -r$

$\frac{1}{3}t + \frac{10}{3}y + \frac{8}{3}z - 11 = -s$

$\frac{1}{3}t + \frac{4}{3}y - \frac{1}{3}z - 2 = -x$

& $x, y, z, r, s, t \geq 0$

From 2nd equat^{n} solve z

$$z = \frac{-s - \frac{1}{3}t - \frac{10}{3}y + 11}{8/3} = -\frac{3}{8}s - \frac{1}{8}t - \frac{10}{8}y + \frac{33}{8} = -z$$

② $-r = \frac{-7}{8}t - \frac{15}{4}y - \frac{5}{8}s - \frac{9}{8}$

③ $-x = \frac{3}{8}t + \frac{7}{4}y + \frac{1}{8}s - \frac{27}{8}$

$$u = \frac{-7}{4}t - \frac{25}{2}y - \frac{5}{4}s + \frac{87}{4}.$$

<u>Exple 2:</u> Solve max $\mu = 2x + 3y$

s.t. $-2x + 3y \leq 2$

$3x + 2y \leq 5$

& $x, y \geq 0$

<u>Solut^{e}:</u> ① Add slack variables & write LPP in standard slack form one has:

$$\boxed{\begin{array}{ll} \max & \mu = 2x + 3y \\ \text{s.t.} & -2x + 3y - 2 = -r \\ & 3x + 2y - 5 = -s \\ \& & x, y, r, s \geq 0 \end{array}}$$

② Establish the simplex tableau

$$\begin{array}{cc|c|l} x & y & 1 & \\ \hline -2 & 3 & -2 & = -r \\ 3 & 2 & -5 & = -s \\ \hline 2 & 3 & 0 & = u \end{array}$$

$x = y = 0$ & $r = 2$ $s = 5$

$u = 2x + 3y = 0$

this solut° is not optimal

pivot operation

make y basic

$a_{12} = 3$ $a_{22} = 2$

$b = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\frac{b_1}{a_{12}} = \frac{2}{3}$

$\frac{b_2}{a_{22}} = \frac{5}{2}$

$\longrightarrow$

$$\begin{array}{cc|c|l} x & r & 1 & \\ \hline -2/3 & 1/3 & -2/3 & = -y \\ 13/3 & -2/3 & -11/3 & = -s \\ \hline 4 & -1 & 2 & = u \end{array}$$

$x = r = 0$ $y = \frac{2}{3}$ $s = \frac{11}{3}$

$u = 4x - r + 2 = 2$

Not optimal

$\downarrow$

$$\begin{array}{cc|c} s & r & 1 \\ \hline 2/13 & 3/13 & -16/13 \\ 3/13 & -2/13 & -11/13 \\ \hline -12/13 & -5/3 & 70/13 \end{array}$$

$s = r = 0$ $x = \frac{11}{13}$ $y = \frac{16}{13}$

$u = \frac{-12}{13} s - \frac{5}{3} r + \frac{70}{13} = \frac{70}{13}$

⚠ optimal solution because all the coefficients are negative

Converting LPP into perfect canonical form: (PCF)

Exple: Maximize $\mu = 4x + 3y + 2z$

s.t. $x + 2y - z + r \leq 3$

$A_2 - 2x + y - s \leq +5$

$A_1 + 4x + 2y + 3z = 7$

& $x, y, z \geq 0$

$r, s \geq 0$

$A_1 \geq 0$

$A_2 \geq 0$

A_1 & A_2 are artificial variables

Exple 2: Establish the simplex tableau

Max $\mu = x + y + s$

s.t.

$x + y + r = 9 \rightarrow r = 9 - x - y$

$2x + y + s = 10 \rightarrow s = 10 - 2x - y$

$4x - 2y + t = 11$

& $x, y, r, s, t \geq 0$

so $\mu = 19 - 2x - 2y$