MATH 404

by Yong Zheng

- linear programming → ch 9
- finite-state Markov chain → ch 10

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2 midterms	200 pts	} Grader	
5 HWs	100 pts		
10 Quizzes	200 pts	→ Myself	
1 Final	200 pts	→ teacher	

Review:

Solve a system of Linear System of Equat^{k}

Gauss eliminat$^{\circ}$ method

Definit$^{\circ}$: (linear equat$^{\circ}$) A linear eqt^{e} in $x_1, x_2, \cdot\cdot\, x_n$ is eqt$^{\circ}$ in the following form: $a_1 x_1 + a_2 x_2 + \cdot\cdot + a_n x_n = b$.

(linear cuz exponent of x is 1)

• A system of linear Eqt^{e} is a collect^{e} of LES in the sa variables.

Exple:
$$\begin{cases} 2x_1 - x_2 + \frac{3}{2}x_3 = 8 \\ x_1 \quad\quad - 4x_3 = -7 \end{cases} \qquad (2)$$

• A solut of a system of LES is a list of numbers $(s_1, s_2, \cdots, s_n)$ that makes each eqt$^{\circ}$ a true statem when the values $s_1, s_2, \cdot\&, s_n$ are substituted for x_1, x respectively.

Exple $(5, \frac{13}{2}, 3)$ is a solut$^{\circ}$ of (2)

Matrix notat$^{\circ}$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & & & \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad (3)$$

augmented matrix. $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$

coefficient matrix $= A$ $\qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

<u>Theorem</u>: system (3) has either
- no solut$^{\text{n}}$
- exactly one solut$^{\text{n}}$
- infinitely many solut$^{\text{ns}}$

<u>Definit$^{\text{n}}$</u>: 2 systems are <u>equivalent</u> if they both have the <u>same</u> <u>solut$^{\text{ns}}$</u>.

e.g. $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \Leftrightarrow \begin{cases} (a_{11}+4a_{21})x_1 + (a_{12}+4a_{22})x_2 = b_1 + 4 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

(!) 3 elementary row operat° :

- replacement
- interchange
- scaling

Exple.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

solut°

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \xrightarrow[\text{by [row 3]} + 4\times\text{[row 1]}]{\text{replace [row 3]}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\xrightarrow[\text{by } \frac{1}{2}]{\text{multiply [row 2]}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} \xrightarrow[\text{by [row 3]} + 3\text{[row 2]}]{\text{replace [row 3]}} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 0 \leftarrow$$
$$x_2 - 4x_3 = 4 \leftarrow$$
$$x_3 = 3 \leftarrow$$

$$\Downarrow$$

$$x = \begin{bmatrix} 29 \\ 16 \\ 3 \end{bmatrix}$$

Theorem: A system $Ax = b$ has a solut^e
$\Leftrightarrow$ the echelon form of the augmented matrix does NOT have a row like $[0, 0, 0, \dots, b]$ $b \neq 0$

Exple: choose h & k s.t. the system

$$x_1 + h x_2 = 2$$
$$4x_1 + 8x_2 = k$$

has: • no solut^e
• a uniq solut^e
• many solut^es

solut^n:
$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

• if $8-4h = 0$ & $k-8 \neq 0$, i.e $h=2$ $k \neq 8$
then the system doesn't have a solut^e
• if $8-4h \neq 0$, i.e. $h \neq 2$,
then the system has a uniq solut^e
• if $8-4h=0$ & $k-8=0$, i.e. $h=2$ & $k=8$,
then the system has infinitely many solut^s.

Cramer's rule: $Ax = b$
1. # of unknown variables = # of eq^ts
(coefficient matrix is a square matrix)
2. $\det A \neq 0$ $\quad x_i = \dfrac{\det A_i}{\det A}$ | A_i = matrix formed when we replace the ith column by b

Exple

$$\begin{cases} 2x_1 + x_2 = 5 \\ x_1 + x_2 = 3 \end{cases} \qquad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \det A = 1$$

$$A_1 = \begin{bmatrix} 5 & 1 \\ 3 & 1 \end{bmatrix} \Rightarrow \det A_1 = 2$$

$$A_2 = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \Rightarrow \det A_2 = 1$$

$$x_1 = \frac{\det A_1}{\det A} = \frac{2}{1} = 2$$

$$x_2 = \frac{\det A_2}{\det A} = \frac{1}{1} = 1.$$

Hw sect° 1.1 : 14.16
s sect° 1.2 : 20

Sect$^{\circ}$ 1.5: Homogeneous linear systems.

Def.: A system of linear eqt^{es} is homogeneous if it can be written as $A\,x = 0$

$A : m \times n$

$\underline{x} = n \times 1$

$\underline{0} = m \times 1$

$Ax = 0$ must have at least 1 solut^{e} $x = \bar{0}$
↳ trivial solut^{e}.

We are going to try to find a non-trivial solut^{e} to $Ax = 0$
↳ not all $x_1, \dots, x_n$ are 0.

Theorem: $Ax = 0$ has a non-trivial solut^{e} iff it has at least 1 free variable

e.g.: $\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$

$$Ax = 0, \quad A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 3 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} 3x_1 - 4x_3 = 0 \\ x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$\begin{cases} Ax = b \\ Ay = 0 \end{cases}$$

$$\Rightarrow A(x+y) = b$$

$$Ax = b$$

$$\cancel{x}y = x_H + p$$

x_H: Homogeneous solut$^{=}$

p: Specific solut^{e}

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -5 & h+4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & h-6 \end{pmatrix}$$

$$\underline{h = 6}$$

e.g. $Ax = b$

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \qquad b = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

$$x_H = \begin{pmatrix} 4/3\, x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$[A \mid b] = \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3x_1 - 4x_3 = -3$$

$$x_2 = 2$$

$$\begin{cases} x_1 = \frac{4}{3} x_3 - 1 \\ x_2 = 2 \end{cases} \Rightarrow p = \begin{pmatrix} 4/3\, x_3 - 1 \\ 2 \\ x_3 \end{pmatrix}$$

set $x_3 = 0, 1, 2, 3, \ldots$

for $x_3 = 1 \to p = \begin{pmatrix} 1/3 \\ \ldots \end{pmatrix}$

So the general solut$^{°}$ here of $Ax = b$ is:

$$x_g = \underbrace{x_3 \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}}_{x_H \text{ (don't set } x_3 \text{ here)}} + \underbrace{\begin{pmatrix} 1/3 \\ 2 \\ 1 \end{pmatrix}}_{\text{set } x_3}$$

Theorem 6: suppose $Ax = b$ is consistent for some b & let p be a solut$^{°}$ $\Rightarrow x_g = x_H + p$

Sect$^{°}$ 1.7:

- Linearly independent:

  Vectors $S = \{ \underline{v_1}, \underline{v_2}, \dots \underline{v_p} \}$ in $\mathbb{R}^n$ $\quad \left(v_i = \begin{pmatrix} x \\ x \\ \vdots \end{pmatrix}_n \right.$

  are linearly independent if

  $x_1 \underline{v_1} + x_2 \underline{v_2} + \dots + x_p \underline{v_p} = 0$ has only the trivial solut$^{°}$.

  as in $x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \bar{0}$ (doesn't have the non-trivial sol

- Linearly dependent:

  if there exist weights $c_1, c_2, \dots, c_p$ not all 0

  s.t. $c_1 \underline{v_1} + c_2 \underline{v_2} + \dots + c_p \underline{v_p} = 0$

  (has non-trivial solut$^{°}$

$$x_1 \bar{v_1} + x_2 \bar{v_2} + \dots + x_p \bar{v_p} = 0 \Leftrightarrow (\bar{v_1}, \bar{v_2}, \dots, \bar{v_p}) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = 0$$

$$\Leftrightarrow A_{n \times p} \cdot x = 0$$

if $Ax = 0$ $\begin{cases} \text{has non-trivial solut}^{°} \Rightarrow \text{dependent} \rightarrow \text{all columns vectors in } A \\ \text{doesn't have a non-trivial solut}^{°} \Rightarrow \text{independent} \end{cases}$

$c = \frac{m}{n}$

$a \nmid$

$\wedge . b$

linearly dependent

$\Uparrow$

free variable(s) $\Rightarrow$ Non-trivial solution   e.g. $x = \begin{pmatrix} 11x \\ x_3 \\ x_3 \end{pmatrix}$

NB Anita

e.g. $A = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 0 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{pmatrix}$

No free variable $\Rightarrow$ independent

$c = \frac{m}{n}$

$a \nmid q$

$b \nmid q$

$ab \nmid q$

$ab \nmid q$

theorem 8: if a set contains more vectors than the # of entries in each vector $\Rightarrow$ set is linearly dependent.

$S = \{v_1, v_2, \dots v_p\}$ in $\mathbb{R}^n$

if $p > n \Rightarrow S$ are dependent.

$\frac{1}{4}$) -2 $/8$

Theorem 9: if $S = \{\bar{v}_1, \bar{v}_2 \dots \bar{v}_p\}$ in $\mathbb{R}^n$ contains a $\bar{0}$ vector $\Rightarrow$ set are linearly dependent

proof: $v_1, v_2, \dots v_p$

if $v_1 = 0$, $1 \cdot 0 + 0 \cdot v_2 + 0 \cdot v_3 + \dots + 0 \cdot v_p = 0$

$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

x^2

e.g. $\begin{pmatrix} -4 & -3 & 0 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \\ 0 & -1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & -3 & 12 \\ 0 & -1 & 4 \\ 0 & +1 & 24 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 28 \\ 0 & 0 & 84 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 28 \\ 0 & 0 & 84 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 24 \\ 0 & 0 & 1 \end{pmatrix}$ linearly independent

0 0 0

trivial solution

19/08/09 HW due sep 17 together w/ 1rst HW:

sec 1.5: 3, 5

sec 1.7: 5

sec 1.8: 4

sec 1.9: 15

sec 2.2: 2, 3

V

Sect° 1.8:

Def: transformat° T

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

domain of T ... range of T

e.g.: $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$ $a = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$b = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$

$c = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

by $T(\bar{x}) = A\bar{x}$

(a) find $T(u)$.

$T(u) = Au = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$

(b) find an $\bar{x}$ in $\mathbb{R}^2$ whose image under T is $\bar{b}$.

find $\bar{x}$ s.t. $T(\bar{x}) = \bar{b} = A\bar{x}$

so $\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right)$

so $\bar{x} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$

(c) is there more than one $\bar{x}$ whose image under T is $\bar{b}$?

No. the solut° is uniq according to (b).

(d) Determine if $\bar{c}$ is the range of T.

see if there is a solut $\bar{x}$ for $T(\bar{x}) = \bar{c}$

$$A\bar{x} = \bar{c}$$

$$\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$[A \mid c] = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2\tfrac{1}{2} \end{pmatrix} \rightarrow 0x_1 + 0x_2 = 2\tfrac{1}{2}$$

$$(0 = 2\tfrac{1}{2})$$

so there is no solut

Therefore $\bar{c}$ is not in the range of T.

<u>Linear Transformat^e</u>:

T: if $\begin{cases} (1)\ T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}) & , \bar{u} \,\&\, \bar{v} \text{ in Domain of:} \\ (2)\ T(c.\bar{u}) = c.T(\bar{u}) & , c \text{ is a scalar.} \end{cases}$

$\Rightarrow T$ is linear

e.g: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^1$

$$T(\bar{x}) = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \bar{x}_1$$

is T linear?

select $\bar{u}, \bar{v}$ in $\mathbb{R}^2$

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$L.H.S. = T(\bar{u}+\bar{v}) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = u_1 + v_1$$

$$R.H.S. = T(\bar{u}) + T(\bar{v}) = T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \bar{u}_1 + \bar{v}_1$$

L.H.S = Left Hand Side

(2) $T(c.\bar{u}) = T\left(c\begin{pmatrix}u_1\\u_2\end{pmatrix}\right) = T\begin{pmatrix}cu_1\\cu_2\end{pmatrix} = cu_1$

R.H.S $= cT(\bar{u}) = c.T\begin{pmatrix}u_1\\u_2\end{pmatrix} = c.u_1$

If T is linear,
then (1) $T(\bar{0}) = \bar{0}$
(2) $T(c.\bar{u} + d.\bar{v}) = cT(\bar{u}) + dT(\bar{v})$
(3) $T(c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_p\bar{u}_p) = c_1T(\bar{u}_1) + c_2T(\bar{u}$
$\cdots + c_pT(\bar{u}_p$
$c_1, \cdots c_p$ scalars

proof:
(1) $T(\bar{0}) = T(\bar{0}+\bar{0})$
$= T(\bar{0}) + T(\bar{0}) \Rightarrow T(\bar{0}) = \bar{0}$
(2) $T(c\bar{u} + d\bar{v}) = T(c\bar{u}) + T(d\bar{v})$
$= cT(\bar{u}) + dT(\bar{v})$

e.g.: $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\bar{x}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-x_2\\x_1\end{pmatrix}$
find the image under T of $\bar{u} = \begin{pmatrix}4\\1\end{pmatrix}$, $\bar{v} = \begin{pmatrix}2\\3\end{pmatrix}$, $\bar{u}+\bar{v} = ($

Ans. $T(\bar{u}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$

$T(\bar{v}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}$

$T(\bar{u}+\bar{v}) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}-4\\6\end{pmatrix}$

Sect 1.9:

e.g. $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

suppose T is linear transform from $\mathbb{R}^2$ to $\mathbb{R}^3$

s.t. $T(e_1) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}$ $T(e_2) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$

find a formula for the image of an arbitrary $\bar{x}$ in $\mathbb{R}^2$

set $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Ans: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$

$= x_1 e_1 + x_2 e_2$

$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = T(x_1 e_1 + x_2 e_2)$

$= x_1 T(e_1) + x_2 T(e_2)$

$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformat.

Then there exist a uniq A s.t. $T(\bar{x}) = A\bar{x}$ for all $\bar{x}$ in $\mathbb{R}^n$.

e.g.: find all missing #s

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{pmatrix}$$

Sect 2.2: Inverse Matrix

An $n \times n$ matrix is invertible there exist $n \times n$ C

s.t. $CA = I$

$AC = I$

$\Rightarrow C = A^{-1}$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Theorem 4: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(i) if $ad - bc \neq 0$ (det A), then A is invertible $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(ii) if $ad - bc = 0$

$\Rightarrow A$ is not invertible

Theorem 5: If A is invertible n then for any $\bar{b}$ in $\mathbb{R}^n \Rightarrow A\bar{x} =$ have a uniq solut $\bar{x} = A^{-1}$.

Property: A, B invertible

(i) A^{-1} invertible / $(A^{-1})^{-1} =$

(ii) AB is invertible

$(AB)^{-1} = B^{-1}A^{-1}$

check: $(AB) \cdot B^{-1}A^{-1} = A(B \cdot B^{-1})A^{-1} = AA^{-1}$

$-40-(-35) = -40+35$
$= -5$

iii) A^T is invertible $\Rightarrow (A^T)^{-1} = (A^{-1})^T$
(check: $A^T \cdot (A^{-1})^T = (A^{-1}A)^T = I^T = I$)

⚠ An Algorithm A^{-1}

$$[A \,\vdots\, I] \xrightarrow[\text{operates}]{\text{row}} [I \,\vdots\, A^{-1}]$$

e.g. $A = \begin{pmatrix} 5 & 10 \\ 4 & 7 \end{pmatrix}$

$$[A \,\vdots\, I] \leftrightarrow \left(\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right)$$

$\frac{1}{5} - \frac{40}{5}$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4 & -1 \end{array}\right)$$

$35 - 40 = -5$

$-\frac{1}{5}\begin{pmatrix} 7 & -10 \\ -4 & 5 \end{pmatrix}$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -39/5 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right)$$

$$\cancel{A^{-1} = \begin{bmatrix} -39/5 & 2 \\ 4 & -1 \end{bmatrix}}$$

$$A^{-1} = \begin{pmatrix} -7/5 & 2 \\ 4/5 & -1 \end{pmatrix}$$

VBA

Linear Programming Problem (LPP) — Geometric method

example 1: A car manufacturer makes 2 cars

	Assembly	Finishing	Profit
Model A	4 hours	6 hours	400
Model B	6 hours	3 hours	300

Assume it has 720 hours of assembly time & 480 hours of finishing time available

Q: How many of each Model will it make s.t. it maximizes its profit

Solut: Let x_1 be the number of unit of Model A to be made
x_2 . . . of Model B . . .

- Total Profit: $400x_1 + 300x_2$
  (we need to maximize this ↑)
- constraints

$$4x_1 + 6x_2 \leq 720$$
$$6x_1 + 3x_2 \leq 480$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

objective fct^e

Maximize $400x_1 + 300x_2$ subject to $4x_1 + 6x_2 \leq 720$

$6x_1 + 3x_2 \leq 480$

& $x_1 \geq 0, x_2 \geq 0$

Canonical form of LPP:

Find $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} \in \mathbb{R}^n$ to MAXIMIZE $f(\bar{x}) = c_1x_1 + c_2x_2 + \dots + c_nx_n$

with the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_n$$

& $x_j \geq 0 \quad j = 1, 2, \dots n$

Matrix notat^e:

$$C = \begin{bmatrix} c_1 \\ c_2 \\ c_n \end{bmatrix} \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$$

Maximize $f(\bar{x}) = C^T\bar{x} \rightarrow$ objective fct^e

subject to $A\bar{x} \leq b$ & $\bar{x} \geq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

HW: sect^{n} 9.2
2, 4, 7, 8
Due sep 17

<u>Definit^{n}</u>: (feasible solut^{n} set) $\mathcal{F}$

$$\mathcal{F} = \{ x \in \mathbb{R}^n : Ax \le b \ \& \ \bar{x} \ge 0 \}$$

<u>Exple</u>: what's the feasible solut^{n} set of LPP in exple 1?

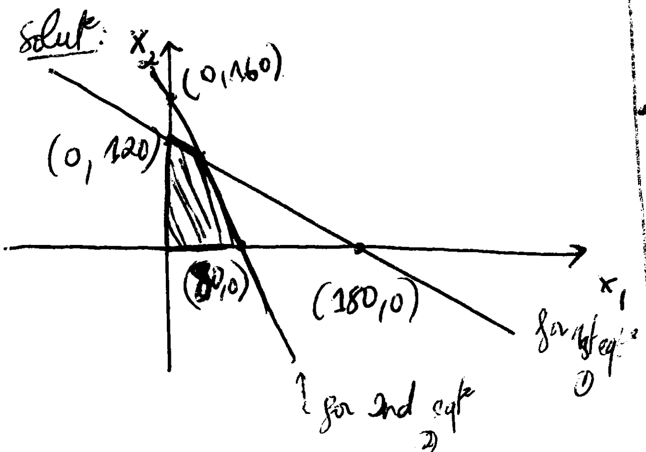
Maximize $400x_1 + 300x_2$

subject to $4x_1 + 6x_2 \le 720$ ①

$6x_1 + 3x_2 \le 480$ ②

& $x_1 \ge 0$ & $x_2 \ge 0$.

<u>Solut^{n}</u>:



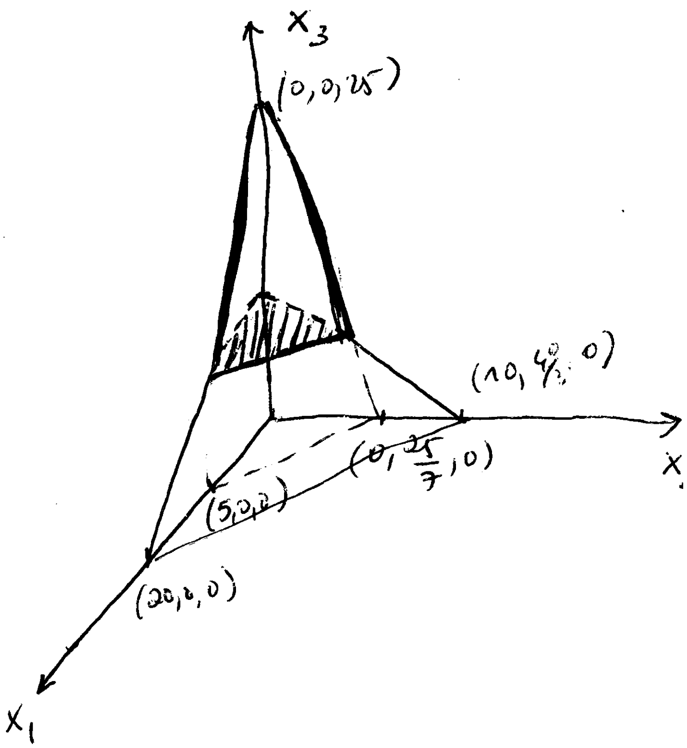
<u>Exple</u>: Find the feasible solut^{n} set of LPP

minimize $3x_1 + x_2 + 5x_3$

subject to $5x_1 + 7x_2 + x_3 \le 25$

$2x_1 + 3x_2 + 4x_3 = 40$

& $x_1 \ge 0$, $x_2 \ge 0$.



- convert minimize $f(\bar{x}) = c^T x$ into maximize $-f(\bar{x}) = -c^T \bar{x}$
- Convert $d^T x \ge s$ into $-d^T x \le -s$
- convert $d^T x = s$ into $d^T x \le s$, $-d^T x \le -s$

<u>Theorem</u>: For a given coronical LPP, if $\mathcal{F} \ne \phi$ & objective fct^{n} is bounded ~~on~~ on $\mathcal{F}$, then LPP has at least one optimal solut^{n}.

Futhermore, at least one optimal solut^{n},

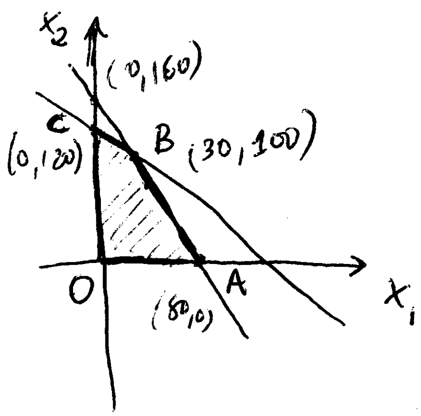
⚠ 1) - Write LPP into canonical form
2) -
3) - picture
4) - find the max value of objective ~~fct~~ & the ~~opti~~ corresponding pt is the optimum

the vertex of $\mathcal{F}$

Example: Solve the LPP in exple 1.

Maximize $400x_1 + 300x_2$
subject to $4x_1 + 6x_2 \leq 720$
$6x_1 + 3x_2 \leq 480$
& $x_1 \geq 0$, $x_2 \geq 0$

solut^{e}:



$\mathcal{F} \neq \phi$

the objective fct^{e} is bounded $\Rightarrow$ it has an optimal solut^{e}.

$$400x_1 + 300x_2 = \begin{cases} 0 & \text{if } (x_1, x_2) = (0,0) \\ 32\,000 & \text{if } (x_1, x_2) = (80,0) \\ 42\,000 & \text{if } (x_1, x_2) = (30,100) \\ 36\,000 & \text{if } (x_1, x_2) = (0,120) \end{cases}$$

Thus $(x_1, x_2) = (30, 100)$ is one optimal solut^{e}

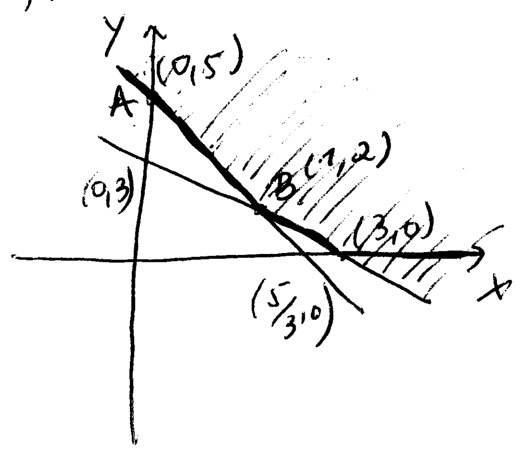
Exple: Minimize $3x + 2y$
subject to $3x + y \geq 5$
$x + y \geq 3$
& $x \geq 0$, $y \geq 0$

solut^{e}:

1) Convert LPP into Canonical fo

Maximize $-3x - 2y$
subject to $-3x - y \leq -5$
$-x - y \leq -3$
& $x \geq 0$, $y \geq 0$.

2) Picture



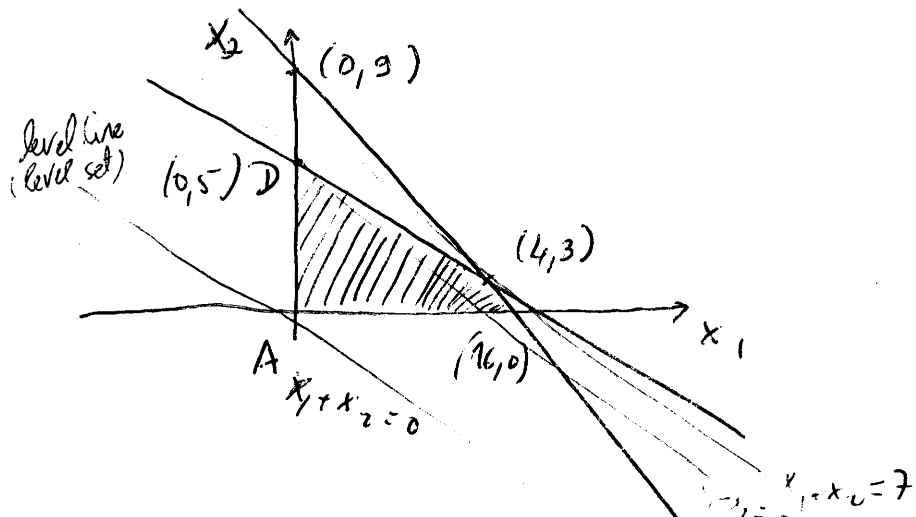
vertex	value of objective fct^{e}	
(0,5)	-10	
(1,2)	-7	optimal solu
(3,0)	-9	

Hw2 sect^e 9.3
9, 10

~~Due sept 17.~~

Geometric method:

Expl: Maximize $x_1 + x_2$
subject to $x_1 + 2x_2 \leq 10$
$3x_1 + 2x_2 \leq 18$
$x_1 \geq 0$, $x_2 \geq 0$



Solut^e: the feasible solut^e set is the shaded area

Coordinates	value of OF (Objective Fct^e) ($x_1 +$
A=(0,0)	0
B = (6,0)	6
C = (4,3)	7 ←
D = (0,5)	5

⚠ (the corresponding lines are parallel.)

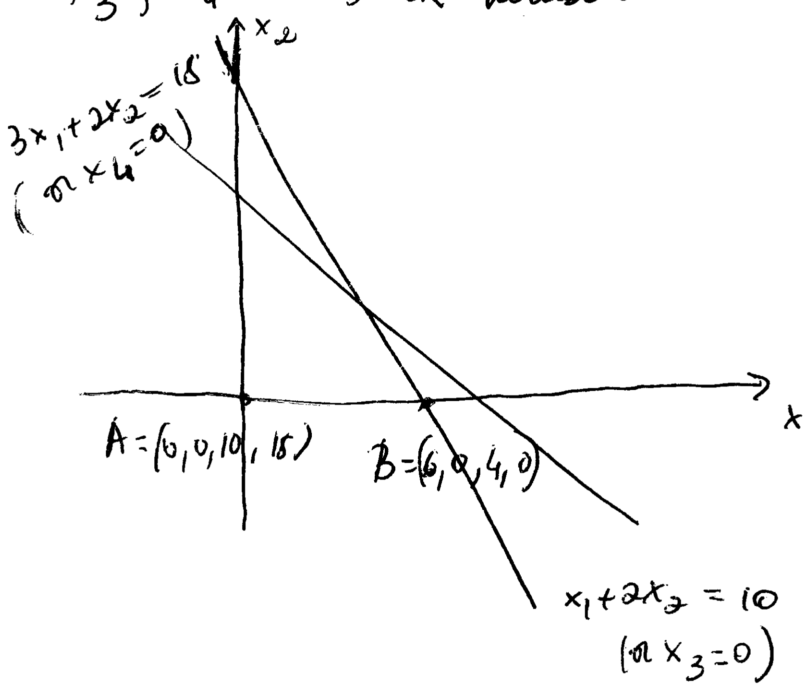
Thus (4,3) is an optimal solut^e.

⚠ **Slack variables:** (sect° 9.3)

Add new variables to constraints & convert them to a system Linear Equat° (LE):

$$x_1 + 2x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + x_4 = 18$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0$$

x_3, x_4 are slack variables



$$x_3 \geq 0 \Rightarrow x_1 + 2x_2 \leq 10$$
$$x_4 \geq 0 \Rightarrow 3x_1 + 2x_2 \leq 18$$

Feasible solut° set: $\{(x_1, x_2, x_3, x_4) ; x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0\}$

e.g.: Solve LPP:

maximize $u = 4x + 3y$

subject to $x + y \leq 4$

$-x + y \leq 2$

& $x \geq 0,\ y \geq 0$

Solut: Add slack variables to constraints & convert them to a system of LE.

$$x + y + s = 4$$
$$x + y + r = 2$$
$$x \geq 0,\ y \geq 0,\ s \geq 0,\ r \geq 0$$

pts		$u =$
$(0,0,4,2)$	: vertex of feasible solu set	0
$(0,4,0,-2)$	(not a vertex)	(not a vertex)
$(0,2,2,0)$		6
$\boxed{(4,0,0,6)}$		$\boxed{16}$ ← optimal pt is $(4,0,$
$(-2,0,6,0)$		(not a vertex)
$(1,3,0,0)$		13

Some definit^{ns}:

Maximize $f(x) = C^T X$

s.t. $Ax \leq b$

& $X \geq 0$

where $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

A $m \times n$ matrix $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

- standard slack form of LPP:

Exple: Max $u = 3x + 4y$

s.t. $x + 2y \leq 12$

$2x - 3y \leq 18$

& $x, y \geq 0$

Add slack variables it becomes

Max $u = 3x + 4y$

s.t. $x + 2y + s = 12$

$2x - 3y + r = 18$

& $x, y, r, s \geq 0$

write this system in the following form:

Max $u = 3x + 4y$

s.t. $x + 2y - 12 = -s$

$2x - 3y - 18 = -r$

& $x, y, s, r \geq 0$

$\rightarrow$ this is a standard slack form.

△ s, r are basic variable
x, y are nonbasic variables

- Canonical form of LPP for Simplex method:
  1) the objective fct is to be maximized
  2) In each equat^e, there is a variable only contained in the eq^{te} & its coefficient is 1.

- Basic variables & nonbasic variables:
  Assume nonbasic variables equal to 0 & solve the basic variables
  Then we have a solut^e of constraint (system of LEs)
  we call this solut^e "basic point."
  If all entries of a basic point are nonnegative, then we call it basic feasible solut^e.

Theorem:
If LPP has optimal solut^e, then one of the optimal solut^e is at least one basic feasible solut^e.

5. Perfect canonical form of LPP for simplex method:
- LPP is in canonical form
- the constant term ≥ 0
- the objective fct^e is expressed in terms of NONBASIC variables.

Exple. Maximize $u = 2x + y$

s.t. $x + 2y + s = 12$

$2x - y + 3r + t = \boxed{-19}$

& $x, y, r, s, t \geq 0$

[This ~~is not~~ in perfect canonical form cuz the a constant term < 0 (-19).]

(* we try to convert it into a perfect canonical form*)

* pivot operat°

Example: Maximize $u = 4x - 3y + 2z$

s.t. $2x + y + z \leq 12$

$-x + 2y + 3z \leq 9$

$3x + 4y - z \leq 6$

& $x, y, z \geq 0$

Add slack variables, this becomes

Max $u = 4x - 3y + 2z$

s.t. $2x + y + z + s = 12$

$-x + 2y + 3z + r = 9$

$3x + 4y - z + t = 6$

& $x, y, z, s, r, t \geq 0$

This LPP is in perfect canonical form

Write LPP in standard slack form.

Max $u = 4x - 3y + 2z$

s.t $2x + y + z - 12 = -s$

$-x + 2y + 3z - 9 = -r$

$3x + 4x - z - 6 = -t$

& $x, y, z, s, r, t \geq 0$

$\frac{(1\times 3)}{2}$

$-1 -$

$5/2 = 2 - \frac{1(x-1)}{2}$ $-5/2 = -1 - \frac{(1\times 3)}{2}$

pivot elmt

$$
\begin{array}{l}
\\
\text{eqt} \\
\text{1 eqt} \\
\text{2 eqt} \\
\text{the gol}^{e}
\end{array}
\begin{array}{cccc|cl}
x & y & z & & 1 & \\
\hline
2 & 1 & 1 & & -12 & = -s \\
-1 & 2 & 3 & & -9 & = -r \\
3 & 4 & -1 & & -6 & = -t \\
\hline
4 & -3 & 2 & & 0 & = u
\end{array}
\qquad
\begin{array}{ccc|cl}
s & y & z & 1 & \\
\hline
1/2 & 1/2 & 1/2 & -6 & = -x \\
1/2 & 5/2 & 7/2 & -15 & = -r \\
-3/2 & 5/2 & -5/2 & 12 & = -t \\
\hline
-2 & -5 & 0 & 24 & = u
\end{array}
$$

- replace pivot elmt by its reciprocal
- divide every other entry in the pivot row by the pivot elmt.
- divide every other entry in the pivot column by (- pivot elmt)
- . . .
- Interchange the posit^e of x & s.

1/22/09 [HW online]

Example 1: solve Max $u = 4x - 3y + 2z$

s.t. $2x + y + z \le 12$

$-x + 2y + 3z \le 9$

$3x + 4y - z \le 6$

& $x, y, z \ge 0$

solut^e: ① Add slack variables to constraints

Max $u = 4x - 3y + 2z$

s.t $2x + y + z + r = 12$

$-x + 2y + 3z + s = 9$

$3x + 4y - z + t = 6$

& $x, y, z, r, s, t \ge 0$.

LPP is in perfect canonical for

② write LPP in standard slack form:

$\max\ u = 4x - 3y + 2z$

s.t. $2x + y + z - 12 = -r$

$-x + 2y + 3z - 9 = -s$

$3x + 4y - z - 6 = -t$

w/ $x, y, z, r, s, t \geq 0$

③ establish the simplex tableau

x	y	z	1	
2	1	1	-12	$= -r$
-1	2	3	-9	$= -s$
3	4	-1	-6	$= -t$
4	-3	2	0	$= \mu$

From this tableau, we assume $x = y = z = 0$

then $r = 12,\ s = 9,\ t = 6$, $\mu = 4x - 3y + 2z + 0 =$

this basic feasible solut° is NOT optimal.

pivot operat°

make x basic (since the coef of x is the largest positive number.

$a_{11} = 2 > 0$

$a_{31} = 3 > 0$

$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 6 \end{bmatrix}$

$\frac{b_1}{a_{11}} = \frac{12}{2} = 6$ $\frac{b_3}{a_{31}} = \frac{6}{3} = 6$

⚠ the variable corresponding to the smallest ratio is to be nonbasic.

pivot the circled 3 & the tableau becomes the following

	t	y	z	1	
	$-2/3$	$-5/3$	$5/3$	-8	$= -r$
	$1/3$	$19/3$	$8/3$	-11	$= -s$
	$1/3$	$4/3$	$-1/3$	-2	$= -x$
	$-4/3$	$-25/3$	$10/3$	8	$= u$

$r=8 \quad s = 11 \quad x = 2$

$u = 8 = -\frac{4}{3}t - \frac{25}{3}y + \frac{10}{3}z$

it's not optimal too

Apply pivot operat^{n} one more time
make z basic

$a_{13} = \frac{5}{3}$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 2 \end{bmatrix}$$

$a_{23} = \frac{8}{3}$

$$\frac{b_1}{a_{13}} = \frac{8}{5/3} = \frac{24}{5}$$

$$\frac{b_2}{a_{23}} = \frac{11}{8/3} = \frac{33}{8}$$

$$\frac{b_1}{a_{13}} =$$

pivot the circled $\frac{8}{3}$ the tableau becomes:

	t	y	s	1	
	$-7/8$	$-15/4$	$-5/8$	$-9/8$	$= -r$
	$1/8$	$5/4$	$3/8$	$-33/8$	$= -z$
	$3/8$	$7/4$	$1/8$	$-27/8$	$= -x$
	$-7/4$	$-25/2$	$-5/4$	$87/4$	$= u$

$t = y = s = 0 \qquad r = 9/8,\ z = 33/8,\ x = \frac{27}{8}$

$u = 87/4 \qquad$ i.e. $(x, y, z) = (27/8, 0, \frac{23}{8})$ is one optimal solution this is equivalent to doing:

Max $u = -\frac{4}{3}t - \frac{25}{3}y + \frac{10}{3}z + 8$

s.t $\frac{-2}{3}t - \frac{5}{3}y + \frac{5}{3}z - 8 = -r$

$\frac{1}{3}t + \frac{10}{3}y + \frac{8}{3}z - 11 = -s$

$\frac{1}{3}t + \frac{4}{3}y - \frac{1}{3}z - 2 = -x$

& $x, y, z, r, s, t \geq 0$

From 2nd equat^{n} solve z

$$z = \frac{-s - \frac{1}{3}t - \frac{10}{3}y + 11}{8/3} = -\frac{3}{8}s - \frac{1}{8}t - \frac{10}{8}y + \frac{33}{8} = -z$$

(2) $-r = \frac{-7}{8}t - \frac{15}{4}y - \frac{5}{8}s - \frac{9}{8}$

(3) $-x = \frac{3}{8}t + \frac{7}{4}y + \frac{1}{8}s - \frac{27}{8}$

$u = \frac{-7}{4}t - \frac{25}{2}y - \frac{5}{4}s + \frac{87}{4}$.

Exple 2: Solve max $\mu = 2x + 3y$

s.t. $-2x + 3y \leq 2$

$3x + 2y \leq 5$

& $x, y \geq 0$

Solut^{e}: (1) Add slack variables & write LPP in standard slack form one has:

$$\boxed{\begin{array}{ll} \max & \mu = 2x + 3y \\ \text{s.t.} & -2x + 3y - 2 = -r \\ & 3x + 2y - 5 = -s \\ \& & x, y, r, s \geq 0 \end{array}}$$

② Establish the simplex tableau

x	y	1	
-2	3	-2	= -r
3	2	-5	= -s
2	3	0	= u

$x = y = 0$ & $r = 2$ $s = 5$

$u = 2x + 3y = 0$

this solut° is not optimal

pivot operation

make y basic

$a_{12} = 3$ $a_{22} = 2$

$b = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\frac{b_1}{a_{12}} = \frac{2}{3}$

$\frac{b_2}{a_{22}} = \frac{5}{2}$

x	r	1	
-2/3	1/3	-2/3	= -y
13/3	-2/3	-11/3	= -s
4	-1	2	= u

$x = r = 0$ $y = \frac{2}{3}$ $s = \frac{11}{3}$

$u = 4x - r + 2 = 2$

Not optimal

s	r	1
2/13	3/13	-16/13
3/13	-2/13	-11/13
-12/13	-5/3	70/13

$s = r = 0$ $x = \frac{11}{13}$ $y = \frac{16}{13}$

$u = \frac{-12}{13} s - \frac{5}{3} r + \frac{70}{13} = \frac{70}{13}$

⚠ optimal solution because all the coefficients are negative

(PCF)

Converting LPP into perfect canonical form:

Exple: Maximize $\mu = 4x + 3y + 2z$

s.t. $x + 2y - z + r \leq 3$

$A_2 - 2x + y - s \leq +5$

$A_1 + 4x + 2y + 3z = 7$

& $x, y, z \geq 0$

$r, s \geq 0$

$A_1 \geq 0$

$A_2 \geq 0$

A_1 & A_2 are artificial variables

Exple 2: Establish the simplex tableau

Max $\mu = x + y + s$

s.t. $x + y + r = 9 \Rightarrow r = 9 - x - y$

$2x + y + s = 10 \Rightarrow s = 10 - 2x - y$

$4x - 2y + t = 11$

& $x, y, r, s, t \geq 0$ so $\mu = 19 - 2x - 2y$

$A > 0 \Rightarrow MA \to +\infty \Rightarrow M \to +\infty$

9/24/09

Quiz on sept 29

Big M method

Expl 1: solve LPP

$$\boxed{\begin{aligned} &\text{minimize} && u = 3x + 2y \\ &\text{s.t.} && x + 2y \geq 2 \\ &&& 3x - y \geq 4 \\ &\& && x, y \geq 0 \end{aligned}}$$

Solut^{n}: $\max -u = -3x - 2y$

s.t. $x + 2y - r = 2$

$3x - y - s = 4$

& $x, y, r, s \geq 0$

Add artificial variables to constraints

$\max -u = -3x - 2y$

s.t. $x + 2y - r + A = 2$

$3x - y - s + B = 4$

& $x, y, r, s, A, B \geq 0$

If this system has an optimal solut, then both A & B must be zero.

Change our OF into

$\max -u = -3x - 2y - MA - MB$

where M is an extremely large positive number.

$$\boxed{\begin{aligned} &\max && -u = -3x - 2y - MA - MB \\ &\text{s.t.} && x + 2y - r + A = 2 \\ &&& 3x - y - s + B = 4 \\ &\& && x, y, r, s, A, B \geq 0 \end{aligned}}$$

It's not in perfect canonical f cuz the OF contains the basic variable A & B

From 1st eq^{t} $A = 2 - x - 2y + r$

2nd eq^{t} $B = 4 - 3x + y + s$

plug values of A & B into OF:

$-u = -3x - 2y - M(2 - x - 2y + r) - M(4 - 3x + y$

$= (4M - 3)x + (M - 2)y - Mr - Ms - 6$

the system becomes:

$\max -u = (4M-3)x + (M-2)y - Mr - Ms - 6$

s.t. $x + 2y - r + A = 2$

$3x - y - s + B = 4$

& $x, y, r, s, A, B \geq 0$

standard slack form

$\max -u = (4M-3)x + (M-2)y - Mr - Ms - 6$

s.t. $x + 2y - r - 2 = -A$

$3x - y - s - 4 = -B$

& $x, y, r, s, A, B \geq 0$

Establish the simplex tableau

x	y	r	s	1	
1	2	-1	0	-2	$= -A$
3	-1	0	-1	-4	$= -B$
$-3+4M$	$-2+M$	$-M$	$-M$	$-6M$	$= -u$

$\downarrow$

B	y	r	s	1	
$-1/3$	$7/3$	-1	$1/3$	$-2/3$	$= -A$
$1/3$	$-1/3$	0	$-1/3$	$-4/3$	$= -x$
$-\frac{4}{3}M+1$	$-3+\frac{7}{3}M$	$-M$	$-1+\frac{1}{3}M$	$\frac{2}{3}M-4$	$= -u$

$\downarrow$

B	A	r	s	1	
$-1/7$	$3/7$	$-3/7$	$1/7$	$-2/7$	$= -y$
$2/7$	$1/7$	$-1/7$	$-2/7$	$-10/7$	$= -x$
$4/7-M$	$9/7-M$	$-9/7$	$-4/7$	$-34/7$	$= -u$

$B = A = r = s = 0 \qquad x = \frac{10}{7} \quad y = \frac{2}{7}$

$\max \; -u = \frac{-34}{7}$

$(x, y) = \left(\frac{10}{7}, \frac{2}{7}\right)$ is an optimal solution

⚠ the artificial variable is not zero
↳ the original problem is not feasible

Dual problem:

Canonical form of LPP

$$\boxed{\begin{array}{ll} \max & f(x) = c^T x \\ \text{s.t.} & Ax \le b \\ \& & x \ge 0 \end{array}}$$

$\updownarrow$ dual

$$\boxed{\begin{array}{ll} \min & g(y) = b^T y \\ \text{s.t.} & A^T y \ge c \\ \& & y \ge 0 \end{array}}$$

Example: Maximize $u = 5x + 2y$

subject to
$$x + 4y \le 3$$
$$-3x + 2y \le 4$$
$$\& \quad x, y \ge 0$$

$\updownarrow$

Minimize $v = 3a + 4b$

s.t.
$$a - 3b \ge 5$$
$$4a + 2b \ge 2$$
$$\& \quad a, b \ge 0$$

HW sect 2.4
11, 12.

Example: Max $u = 5x + 2y$
s.t. $x + 2y \le 12$
$3x - 4y \le 9$
$7x + 8y \le 20$
& $x, y \ge 0$

$\updownarrow$

Min $v = 12a + 9b + 20c$
s.t. $a + 3b + 7c \ge 5$
$2a - 4b + 8c \ge 2$
& $a, b, c \ge 0$

Max $u = 5x + 2y$
s.t $x + 2y - 12 = -r$
$3x - 4y - 9 = -s$ ①
$7x + 8y - 20 = -t$
& $x, y, r, s, t \ge 0.$

Min $v = 12a + 9b + 20c$
st. $a + 3b + 7c - 5 = d$
$2a - 4b + 8c - 2 = e$ ②
& $a, b, c, d, e \ge 0$

① basic variables

$$\begin{array}{c|cc|c|l} & x & y & 1 & \\ \hline a & 1 & 2 & -12 & = -r \\ b & 3 & -4 & -9 & = -s \\ c & 7 & 8 & -20 & = -t \\ \hline -1 & 5 & 2 & 0 & = u \\ \hline & d & e & -v & \end{array}$$

Canonical form of LPP:
max $f(x) = c^T x$
s.t. $Ax \le b$ (1)
& $x \ge 0$

Definit°: (Dual problem) the following problem
$$\left.\begin{array}{l} \min\ g(y) = b^T y \\ \text{s.t. } A^T y \ge c \\ \quad y \ge 0 \end{array}\right\} \quad ②$$
is called the dual problem of (

Example: max $u = 5x - 2y$
s.t. $3x + y \le 7$
$4x - 2y \le 3$
& $x, y \ge 0$

Dual problem is

$$\begin{aligned} \min \quad & V = 7a + 3b \\ \text{s.t.} \quad & 3a + 4b \geq 5 \\ & a - 2b \geq -2 \\ \& \quad & a, b \geq 0 \end{aligned}$$

Exple 2:

$$\begin{aligned} \max \quad & u = 5x + 2y \\ \text{s.t.} \quad & 4x - y = 3 \\ & x + y \geq 5 \\ \& \quad & x, y \geq 0 \end{aligned}$$

Solut: Rewrite the LPP into canonical form

$$\begin{aligned} \max \quad & u = 5x + 2y \\ \text{s.t.} \quad & 4x - y \leq 3 \\ & -4x + y \leq -3 \\ & -x - y \leq -5 \\ \& \quad & x, y \geq 0 \end{aligned}$$

$\downarrow$ dual problem

$$\begin{aligned} \min \quad & V = 3a - 3b - 5c \\ \text{s.t.} \quad & 4a - 4b - c \geq 5 \\ & -a + b - c \geq 2 \\ \& \quad & a, b, c \geq 0 \end{aligned}$$

② canonical form $\longrightarrow$

$$\begin{aligned} &\max && -g(y) = -b^T y \\ &\text{s.t.} && -A^T y \leq -c \\ &\& && y \geq 0 \end{aligned}$$

$\big\downarrow$ dual problem

$$\boxed{\begin{aligned} &\min && h(z) = -c^T z \\ &\text{s.t.} && -Az \geq -b \\ &\& && z \geq 0 \end{aligned}}$$

- Solve LPP & its dual together

Exple 3: Solve

$$\begin{aligned} &\max && u = -3x + 2y \\ &\text{s.t.} && x + 4y \leq 3 \\ & && 3x - y \leq 5 \\ &\& && x, y \geq 0 \end{aligned}$$

& its dual.

Solut^e: the dual problem is:

$$\begin{aligned} &\min && v = 3a + 5b \\ &\text{s.t.} && a + 3b \geq -3 \\ & && 4a - b \geq 2 \\ &\& && a, b \geq 0. \end{aligned}$$

(sl^n cont'd)

Add some slack variables to the max... (to get it in perfect can

max $\mu = -3x + 2y$

st $x + 4y + r = 3$ → in perfect canonical form

$3x - y + s = 5$ cuz the values (+3, +5) a

& $x, y, r, s \geq 0$ positive, & μ is expr in nonbasic variables (x

~~max v =~~ 3

min $v = 3a + 5b$

s.t. $a + 3b - c = -3$

$4a - b - d = 2$

& $a, b, c, d \geq 0$

Simplex ~~t~~ tableau:

	x	y	1	
a	1	4	-3	$= -r$
b	3	-1	-5	$= -s$
-1	-3	2	0	$= \mu$
	c	d	-v	

→

	x	r	1	
d	1/4	1/4	-3/4	$= -y$
b	13/4	1/4	-17/4	$= -s$
-1	-7/2	-1/2	3/2	$= \mu$
	c	a	-v	

$d = b = 0 \to c = \frac{7}{2}, a = \frac{1}{2}$

min $v = \frac{3}{2}$

optimal sol^n of minimal problem is ←

$\boxed{(a, b) = (\frac{1}{2}, 0) \quad \text{min } v = 3/2}$

$x = r = 0 \to y = 3/4, s = \frac{17}{4}$

max $\mu = 3/2$

optimal sol^n of max problem $\boxed{(x, y) = (0, 3/4) \quad \text{max } \mu = 3/2}$

9.3: 9, 10
9.4 11, 12

HW set
9.1:
, 12, 14

Matrix game:

2 player & zero-sum

Ex 1: player R picks number 1, 2 or 3
player C guesses at the number player R picked

rule: if player C guesses correctly then he wins \$1
otherwise he loses \$2

player C → loser player

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \end{array}$$

player R ↓ winner player

Rock, scissors & paper:
scissors beats paper
Rock beats scissors
paper beats rock

winner player

$$\begin{array}{cc} & \begin{array}{ccc} R & S & P \end{array} \\ \begin{array}{c} R \\ S \\ P \end{array} & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

→ loser player.

We don't know how to control this game
bcz there's no saddle pt.

Assumpt^{es}:

(I) players are rational

they play to win the most or lose the least

(II) players are conservative

• the goal of now player (winner player) is to ~~max~~ ~~minimize the~~ maxcimize the mi

• the goal of column player (loser player) is to min (max val

explе:

Saddle ~~center~~ pt of this game

$$
\begin{array}{r}
\\ \\ \\ \\ \text{max}
\end{array}
\begin{array}{c}
\left[\begin{array}{ccc} 4 & -2 & 3 \\ -8 & -3 & 5 \\ 5 & \boxed{-1} & 2 \end{array}\right] \\
\begin{array}{ccc} 5 & -1 & 5 \end{array}
\end{array}
\begin{array}{c}
\text{min} \\ -2 \\ -8 \\ -1 \\ \\
\end{array}
\quad \max(\min) = -1
$$

$\min(\max) = -1$

Expected value of variable:

toss a die.

$S = \{1, 2, 3, 4, 5, 6\}$

$P \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6$

$E(S) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$E(S) = \frac{21}{6}$ → expected value of this radom valuable

(mean value)

$$
\begin{array}{cc|ccc}
 & \text{proba} & 1/2 & 1/4 & 1/4 \\
\text{proba} & & R & S & P \\
\hline
1/3 & R & 0 & 1 & -1 \\
1/3 & S & -1 & 0 & 1 \\
1/3 & P & 1 & -1 & 0
\end{array}
$$

proba

$0 \rightarrow 1/6$

$1 \rightarrow 1/12$

$-1 \rightarrow 1/12$

$-1 \rightarrow 1/6$

$0 \rightarrow 1/12$

$1 \rightarrow 1/12$

$1 \rightarrow 1/6$

$-1 \rightarrow 1/12$

$0 \rightarrow 1/12$

Expected payoff: $= \sum \ldots$ ($\sum$ of the probas)

$= x^T A y$

probability vector

A is a $m \times n$ matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \text{ is probability vector}$$

if $x_i \geq 0$

$\sum_{i=1}^{m} x_i = 1$

<u>strategy space</u>

row player R

$$\underline{X} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \text{ is prob. vector} \right\}$$

column player

$$\underline{Y} = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix} \text{ is a prob. vector} \right\}.$$

Expected payoff x^TAy

Goal of row player R (winner player)

$$\max_{x \in X} \min_{y \in Y} x^TAy$$

Goal of column player C (loser player)

$$\min_{y \in Y} \max_{x \in X} x^TAy$$

~~Let x^TAy~~

Let $y = y_1e_1 + y_2e_2 + \cdots + y_ne_n$

$$x^TAy = y_1x^T\underbrace{Ae_1}_{\text{first column}} + y_2x^T\underbrace{Ae_2}_{\text{2nd column}} + \cdots + y_nx^T\underbrace{Ae_n}_{\text{nth column}}$$

$$= y_1x^Ta_1 + y_2x^Ta_2 + \cdots + y_nx^Ta_n$$

$$\min x^TAy = \min_{y \in Y} \{y_1x^Ta_1 + y_2x^Ta_2 + \cdots + y_nx^Ta_n\}$$

(~~Since the $y_s \geq 0$~~ it's the min of the x^Ta_i, assuming the y in front of the min is 1 & the other y_s are 0).

sat 24
17, 18

Matrix game:

$$
A = \begin{array}{c|ccc|c}
 & & & & \text{min} \\
 & 1 & -2 & 2 & -2 \\
 & -2 & 4 & 1 & -2 \\
 & 3 & 1 & -1 & -1 \\
\text{max} & 3 & 4 & 2 &
\end{array}
$$

$-1 \leftarrow$ maximum

minimum of max$_s$

play the game one time.

Row player case:

- $[1,0,0]\, A \begin{bmatrix}1\\0\\0\end{bmatrix} = 1$
- $[1,0,0]\, A \begin{bmatrix}0\\1\\0\end{bmatrix} = -2$
- $[1,0,0]\, A \begin{bmatrix}0\\0\\1\end{bmatrix} = 2$

$\min_y [1,0,0]\, Ay = -2$ where $y = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\min_y [0,1,0]\, Ay = -2$

$\min_y [0,0,1]\, Ay = -1$

then $\max_x \min_y x^T A y$ Goal of row player,

$x = \begin{bmatrix}1\\0\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1\end{bmatrix}$ & $y = \begin{bmatrix}1\\0\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1\end{bmatrix}$ here it's =

$m \times n$ m n

Goal of column ~~the~~ player C is to
$$\min_y \max_x x^T A y$$

$$\boxed{\begin{array}{l}\text{Goal of row player } R \\ \max_{x \in X} \min_{y \in Y} x^T A y \\ \\ \text{Goal of column player } C \\ \min_{y \in Y} \max_{x \in X} x^T A y\end{array}}$$

$$= \max_{x \in X} \min y_1 x^T A e_1 + y_2 x^T A e_2 + \dots + y_n x^T A e_n$$

$$= \max_{x \in X} \min_{y \in Y} y_1 x^T a_1 + y_2 x^T a_2 + \dots + y_n x^T a_n$$

$$= \boxed{\max_{x \in X} \min_i \{x^T a_i\}}$$

where a_i is the ith column

$\Downarrow$

$\min_{y \in Y} \max_j \{b_j y\}$ where b_j is the j^{th} row of A.

Thm: For a given matrix game
$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y$$
this number is called the value of the game.
Moreover, there exists $\hat{x} \in X$, $\hat{y} \in Y$ s.t.
$$\hat{x}^T A \hat{y} = \max_{x \in X} \min_{y \in Y} x^T A y \quad \left(= \min_{y \in Y} \max_{x \in X} x^T A y \right)$$

$\hat{x}$ is the optimal strategy of R
$\hat{y}$ is the optimal strategy of C.

- How to solve a $2 \times n$ matrix game:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}$$

$$\underline{X} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 . \ \begin{array}{l} x_1, x_2 \geq 0 \\ x_1 + x_2 = 1 \end{array} \right\}$$ $\Rightarrow$ strategy space of row player.

$$\underline{X} = \left\{ \begin{bmatrix} 1-t \\ t \end{bmatrix} \in \mathbb{R}^2 \quad t \in [0,1] \right\}$$

the $(x^T a_i)_s$ correspond to $\boxed{(1-t)\, a_{1i} + t\, a_{2i}}$

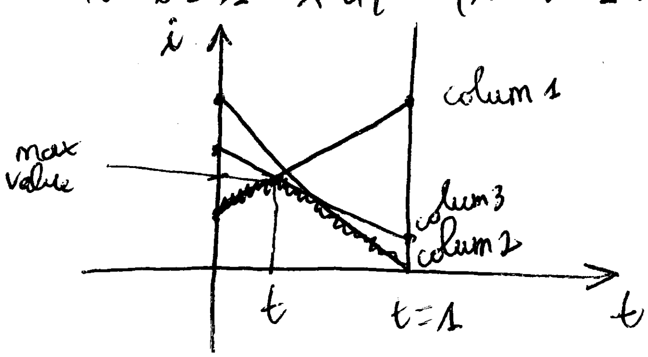
ex: $A = \begin{bmatrix} 1 & 5 & 3 & 6 \\ 4 & 0 & 1 & 2 \end{bmatrix}$

solut: $\underline{X} = \left\{ \begin{bmatrix} 1-t \\ t \end{bmatrix} \in \mathbb{R}^2 \quad t \in [0,1] \right\}$ is the strategy

space of R

$$x^T a_i = (1-t)\, a_{1i} + t\, a_{2i}$$

for $i = 1$ $\quad x^T a_i = (1-t) \cdot 1 + t \cdot 4 = 1 + 3t$



t is intersect of 1rst & 3rd li

So $1 + 3t = 3 - 2t$

$t = \frac{2}{5}$.

opti

$e_1^T A$ = 1rst row of matrix A.

For $i=2$ $\quad x^T a_2 = (1-t)(5) + t \times 0$
$= 5 - 5t$

For $i=3$ $\quad x^T a_3 = (1-t)(3) + (x)$
$= 3 - 2t$

For $i=4$ $\quad x^T a_4 = (1-t)(6) + 2t$
$x^T a_4 = 6 - 4t$

$t = \frac{2}{5}$. the optimal strategy of $R = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} = \hat{x}$ as $t = \frac{2}{5}$

the value of the game is $1 + 3 \times \frac{2}{5} = \frac{11}{5}$.

① Formula of optimal strategy of $\hat{y}$:

$$\hat{y} = y_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$y_1 + y_3 = 1$ $\qquad = y_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

which line passes through the maximum pt, then which $y_i \neq 0$

② Establish a system of linear eqt² of $\hat{y}$.

$\hat{y}$ is a probability vector $\Rightarrow \sum y_i = 1$

$\hat{x} = 3/5\, e_1 + \frac{2}{5} e_2$ $\qquad e_1^T A \hat{y} = \frac{11}{5}$

$e_2^T A \hat{y} = \frac{11}{5}$

$$[1 \quad 0]\, A \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix} = \frac{11}{5}$$

$$\rightarrow [1 \quad 5 \quad 3 \quad 6] \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix} = \boxed{y_1 + 3y_3 = \frac{11}{5}} \quad (1)$$

$$[0, 1]\,[A] \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix} = \frac{11}{5}$$

$$\rightarrow [4 \quad 0 \quad 1 \quad 2] \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix} = \frac{11}{5}$$

$$\boxed{4y_1 + y_3 = \frac{11}{5}} \quad (2)$$

we Know $\boxed{y_1 + y_3 = 1}$ (3)

solve the 3 eqt^{s}:

$y_3 = 3/5$

$y_1 = \frac{2}{5}$

$$\boxed{\hat{y} = \begin{bmatrix} 2/5 \\ 0 \\ 3/5 \\ 0 \end{bmatrix}}$$

Exple:

saddle pt

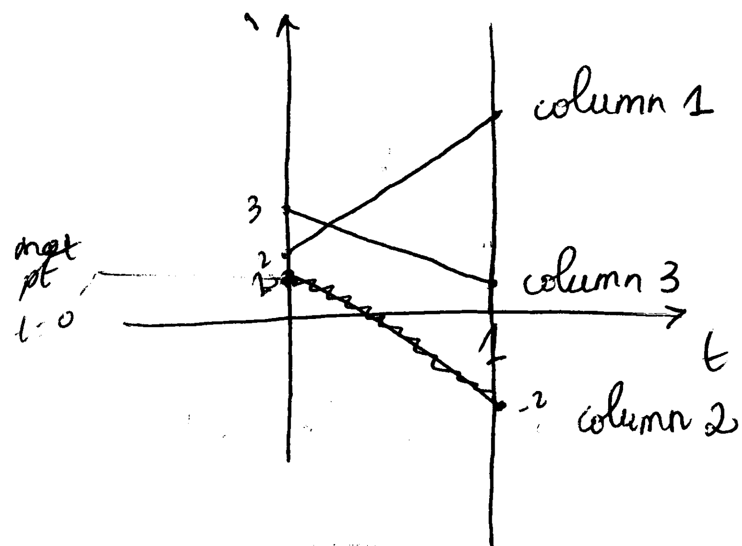
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$$

optimal strategy of R

$\cancel{\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

optimal strategy of C

$$\hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



optimal $\boxed{\hat{x} = \begin{bmatrix} 1-t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

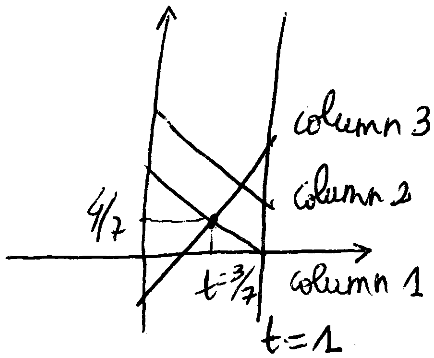
$\hat{y} = y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

~~$y_2 = 1$~~

$\boxed{\hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

Exple:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$



$$[1-t \;\; t]\, A \begin{bmatrix}1\\0\\0\end{bmatrix} = [1-t \;\; t]\begin{bmatrix}1\\0\end{bmatrix} = 1-t$$

$$[1-t \;\; t]\, A \begin{bmatrix}0\\1\\0\end{bmatrix} = [1-t \;\; t]\begin{bmatrix}2\\1\end{bmatrix} = 2-t$$

$$[1-t \;\; t]\, A \begin{bmatrix}0\\0\\1\end{bmatrix} = [1-t \;\; t]\begin{bmatrix}-2\\4\end{bmatrix} = -2+6t$$

$$1-t = -2+6t \Rightarrow t = 3/7$$

optimal strategy of R is $\hat{x} = \begin{bmatrix}4/7\\3/7\end{bmatrix}$.

the value of the game $v = 4/7$

$$\hat{y} = y_1\begin{bmatrix}1\\0\\0\end{bmatrix} + y_2\begin{bmatrix}0\\1\\0\end{bmatrix} + y_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\underset{4/7}{\overset{\shortparallel}{v}} = \hat{x} A \hat{y} = y_1 \underbrace{\hat{x}^T A \begin{bmatrix}1\\0\\0\end{bmatrix}}_{4/7} + \cancel{y_2} \underbrace{\hat{x}^T A \begin{bmatrix}0\\1\\0\end{bmatrix}}_{11/7} + y_3 \underbrace{\hat{x}^T A \begin{bmatrix}0\\0\\1\end{bmatrix}}_{4/7}$$

$$\Rightarrow \hat{y} = \begin{bmatrix}y_1\\0\\y_3\end{bmatrix}$$

Establish a system of linear eqt^{ns}

$y_1 + y_2 = 1$ since $\hat{y}$ is a probability vector

$\frac{4}{7} = v = \hat{x}^T A \hat{y}$.

$$\hat{x} = 4/7\begin{bmatrix}1\\0\end{bmatrix} + 3/7\begin{bmatrix}0\\1\end{bmatrix} = \underline{4/7\, e_1 + 3/7\, e_2}$$

$$\Rightarrow \frac{4}{7} = e_1^T A \hat{y}$$
$$\frac{4}{7} = e_2^T A \hat{y}$$

$$\boxed{\hat{x}^T A \hat{y} = \min_{y \in \bar{Y}} \max_{x \in \bar{X}} x^T A y}$$

$$\Downarrow \quad \hat{x}^T A \hat{y} = \max_{x \in \bar{X}} x^T A \hat{y}$$

$$\geq \begin{cases} e_1^T A y \\ e_2^T A y \end{cases}$$

$$\frac{4}{7} = \hat{x}^T A \hat{y}$$

$$= \frac{4}{7} e_1^T A \hat{y} + \frac{3}{7} e_2^T A \hat{y}$$

$$\leq \frac{4}{7} \hat{x}^T A \hat{y} + \frac{3}{7} \hat{x}^T A \hat{y}$$

$$= \left(\frac{4}{7} + \frac{3}{7}\right) \hat{x}^T A \hat{y} = \hat{x}^T A \hat{y} = \frac{4}{7}$$

so

$$\frac{4}{7} = \frac{4}{7} e_1^T A \hat{y} + \frac{3}{7} e_2^T A \hat{y} \leq \frac{4}{7}$$

~~Goal of R = m~~

Goal of R $= \max_{x \in \bar{X}} \min_i \{x^T a_i\}$

$$x^T \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \leq x^T \cancel{\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}}$$

Greater column is useless
Delete it

Goal of C $= \min_{y \in \bar{Y}} \max_j \{b_j y\}$

where b_j is the jth row of A

$$[1\ 2\ 3\ 4]\, y \leq [2\ 3\ 4\ 5]\, y$$

Small row is useless.
Delete it

Quiz on thursday on matrix games
($2 \times n$)

e.g.: $A = \begin{bmatrix} \cancel{7} & 1 & 6 & 7 \\ \cancel{8} & 3 & 1 & 0 \\ \cancel{4} & 5 & 3 & 3 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 6 & 7 \\ \cancel{3} & \cancel{1} & \cancel{0} \\ 5 & 3 & 3 \end{bmatrix}$

$\Rightarrow C = \begin{bmatrix} 1 & 6 & \cancel{7} \\ 5 & 3 & \cancel{3} \end{bmatrix} \Rightarrow D = \begin{bmatrix} 1 & 6 \\ 5 & 3 \end{bmatrix}$

For D, the optimal strategy of $R = \begin{bmatrix} 2/7 \\ 5/7 \end{bmatrix}$

— " — " — " — C $= \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$

thus $\hat{X} = \begin{bmatrix} 2/7 \\ 0 \\ 5/7 \end{bmatrix}$ $\quad \hat{y} = \begin{bmatrix} 0 \\ 3/7 \\ 4/7 \\ 0 \end{bmatrix}$

A an $m \times n$ matrix game.

Goal of the Row player R (winner player)

$$= \max_{x \in X} \min_{y \in Y} x^T A y = \max_{x \in X} \min_{i} \{x^T a_i\}$$

where a_i is the ith column of A.

↓

If there are 2 columns a_i a_j and $a_i > a_j$, then $X^T a_i \geq x^T a_j$

we cannot find the minimum over i-th column

So the $\boxed{\text{Great column is useless}}$. (delete it)

Goal of column player C (loser player)

$$= \min_{y \in Y} \max_{x \in X} x^T A y = \min_{y \in Y} \max_{j} \{ b_j y \}$$

where b_j is the j-th row of A.

If there are 2 rows b_ℓ, b_s

$b_\ell > b_s$

then $b_\ell y \geq b_s y$

We ~~cannot~~ find the minimum over s-row

$\boxed{\text{Small row is useless}}$ (delete it)

Example 1:

$$A = \begin{bmatrix} -2 & 3 & 0 & 4 \\ 2 & -1 & 3 & 6 \\ 1 & -3 & 4 & 2 \end{bmatrix}$$

Solut: Reduce ~~matrix~~ the payoff matrix (great col. useless & sm row

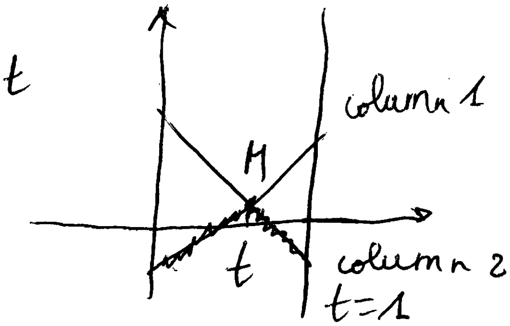
$$A \xrightarrow[\text{col}]{\text{delete 4th}} \begin{bmatrix} -2 & 3 & 0 \\ 2 & -1 & 3 \\ 1 & -3 & 4 \end{bmatrix} \xrightarrow[\text{col.}]{\text{delet 3rd}} \begin{bmatrix} -2 & 3 \\ 2 & -1 \\ 1 & -3 \end{bmatrix} \xrightarrow[\text{3rd row}]{\text{delete}} \begin{bmatrix} -2 & 3 \\ 2 & -1 \end{bmatrix} \triangleq$$

solve a matrix game w/ payoff matrix D.

$X = \left\{ \begin{bmatrix} 1-t \\ t \end{bmatrix} \in \mathbb{R}^2 \quad t \in [0,1] \right\}$ is the strategy space of R

For $i=1$, $[1-t, t]\, D \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -2+4t$

$i=2$ $\quad [1-t, t]\, D \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{2nd col.}} = 3-4t$



solve $-2+4t = 3-4t$

$\Rightarrow t = 5/8$

the optimal strategy of R: $(\hat{x})$

$$\hat{x} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

the value of the game $= -2+4\left(\frac{5}{8}\right) = \frac{1}{2}$

$$\hat{y} = y_1 e_1 + y_2 e_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

D = 1st row
D = 2nd row

$$\begin{cases} y_1 + y_2 = 1 \\ e_1 D \hat{y} = 1/2 \\ e_2 D \hat{y} = 1/2 \end{cases} \Rightarrow \begin{cases} y_1 + y_2 = 1 \\ -2y_1 + 3y_2 = 1/2 \\ 2y_1 - y_2 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} y_1 = 1/2 \\ y_2 = 1/2 \end{cases}$$

$\hat{y} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is the optimal strategy of C.

In the original matrix game A, $\hat{x} = \begin{bmatrix} 3/8 \\ 5/8 \\ 0 \end{bmatrix}$ and $\hat{y} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$

(cuz we deleted the 3rd row; the 3rd & the 4th column).

Ex: ~~let~~ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\hat{x} = \begin{bmatrix} \dfrac{d-c}{a-b-c+d} \\ \dfrac{a-b}{a-b-c+d} \end{bmatrix}$$

b/c $a - at + ct = b - bt + dt$

The value of the game is $= \dfrac{ad+bc}{a-b-c+d}$

$$\hat{y} = \begin{bmatrix} \dfrac{d-b}{a-b-c+d} \\ \dfrac{a-c}{a-b-c+d} \end{bmatrix}$$

For $B = A + \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have the same optimal strate ~~as~~ for A.

But the value of the game is changed & it becomes $= \dfrac{ad - \cancel{bc}}{a-b-c+d} + \alpha$ (V add α,

---

Assume A is a positive matrix. i.e. $a_{ij} \geq 0$; A is $m \times n$

Maximize $u = y_1 + y_2 + \cdots + y_n$

subject to $Ay \leq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $\Longrightarrow$ (gives the optimal solut of the column player)

and $y \geq 0$

$\xrightarrow{\text{dual problem}}$ min $v = x_1 + x_2 + \cdots + x_n$

s.t. $A^T x \geq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

& $x \geq 0$

When solving, we find ~~ŷ is~~ $\bar{y}$ is the optimal solut & $\bar{x}$ is the optimal solut^e.

$$\max u = \min v \triangleq \lambda$$

the optimal strategy of C

Then $\hat{y} = \bar{y}/\lambda$ the optimal strategy of R is $\hat{x} = \bar{x}/\lambda$

The value of the game $= \dfrac{1}{\lambda}$

⚠ If the matrix A has ~~+~~ negative values, we can perturb the matrix by a constant term. $B = A + \alpha \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The optimal strategy of C & R will stay the same; the value of the game will be augmented by α.

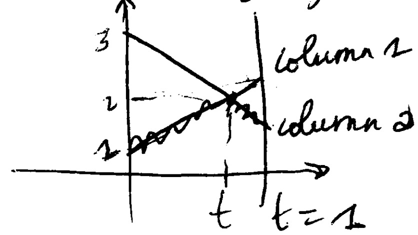
<u>Exple 2</u>: $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

<u>Solut^e</u>: (Geometric method.)

$X = \left\{ \begin{bmatrix} 1-t \\ t \end{bmatrix} \in \mathbb{R}^2 \quad t \in [0,1] \right\}$ is the strategy space

For $i = 1$ $\quad [1-t, t]\, A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + t$

For $i = 2$ $\quad [1-t, t]\, A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 - 2t$



solve $1+t = 3-2t \Rightarrow t = 2/3$

Thus the optimal strategy of R is $\hat{x} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

The value of the game is $\boxed{5/3}$.

Let $\hat{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be the optimal strategy of C.

$$\begin{cases} y_1 + y_2 = 1 \\ y_1 + 3y_2 = 5/3 \\ 2y_1 + y_2 = 5/3 \end{cases} \Rightarrow \begin{cases} y_1 = 2/3 \\ y_2 = 1/3 \end{cases}$$

$\hat{y} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ is the optimal strategy of C

method 2: (LPP)

A is a positive matrix. $\Rightarrow$ so don't need to perturb the matrix

Maximize $\mu = y_1 + y_2$	Minimize $V = x_1 + x_2$
s.t. $y_1 + 3y_2 \leq 1$	s.t. $x_1 + 2x_2 \geq 1$
$2y_1 + y_2 \leq 1$	$3x_1 + x_2 \geq 1$
& $y_1, y_2 \geq 0$	& $x_1, x_2 \geq 0$

$\xrightarrow{\text{dual pb}}$

$\rightarrow$ Add slack variables & write the LPP in standard slack form.

Max $\mu = y_1 + y_2$
s.t. $y_1 + 3y_1 - 1 = r$
$2y_1 + y_2 - 1 = -s$
& $y_1, y_2, r, s \geq 0$

Min $v = x_1 + x_2$
s.t $x_1 + 2x_2 - 1 = a$
$3x_1 + x_2 - 1 = b$
& $x_1, x_2, a, b \geq 0$

→ Establish the simplex tableau:

$$\begin{array}{c|cc|c|l} & y_1 & y_2 & 1 & \\ \hline x_1 & 1 & 3 & -1 & = -r \\ x_2 & 2 & 1 & -1 & = -s \\ \hline -1 & 1 & 1 & 0 & = \mu \\ \hline & a & b & -v & \end{array}$$

→ Apply pivot operat^n twice

$$\begin{array}{c|cc|c|l} & s & r & 1 & \\ \hline b & -1/5 & 2/5 & -1/5 & = -y_2 \\ a & 3/5 & -1/5 & -2/5 & = -y_1 \\ \hline -1 & -2/5 & -1/5 & 3/5 & = \mu \\ \hline & x_2 & x_1 & -V & \end{array}$$

Assume $s = r = 0$ → $y_1 = 2/5$, $y_2 = \frac{1}{5}$ ⇒ $\overline{Y} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix}$ is the optimal solut^e of Max p

Assume $a = b = 0$ ⇒ $x_1 = 1/5$ & $x_2 = 2/5$ ⇒ $\overline{X} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$ is the optim solut^e of the Min pb

Max u = Min $v = 3/5 \stackrel{\Delta}{=} \lambda$. Then $\hat{Y} = \overline{Y}/\lambda = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ & $\hat{X} = \overline{X}/\lambda = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ are opt. strategies of C & R respectively

⚠ the LPP is always in the perfect canonical for the matrix game. ⚠

The value of the game is $= \frac{1}{\lambda} = 5/3$.

Exple:

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

solut:

A has a negative value $\Rightarrow$ perturb A by $2\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 0 & 3 & 4 \\ 5 & 4 & 2 \end{bmatrix}$$

$$\begin{array}{c|ccc|c|l} & r & y_2 & y_3 & 1 & \\ \hline x & 0 & 3 & 4 & -1 & = -r \\ r & 1/5 & 4/5 & 2/5 & -1/5 & = -y_1 \\ \hline & -1/5 & 1/5 & 3/5 & 1/5 & = u \end{array}$$

~~Apply~~ Max $u = y_1 + y_2 + y_3$

s.t. $3y_2 + 4y_3 \leq 1$

$5y_1 + 4y_2 + 2y_3 \leq 1$

& $y_1, y_2, y_3 \geq 0$

dual

Min $v = x_1 + x_2$

s.t. $5x_2 \geq 1$

$3x_1 + 4x_2 \geq 1$

$4x_1 + 2x_2 \geq 1$

& $x_1, x_2 \geq 0$

sect. 9.4 ex 5 → answer

$\lambda = \frac{7}{20}$ $\boxed{\frac{1}{\lambda} = \frac{20}{7}}$

$-1/5$

$\frac{1}{4} \times \frac{20}{7}$

$\boxed{\frac{5}{7} = y_3}$

$\frac{1}{10} \times \frac{20}{7}$

$$\hat{x} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} 2/7 \\ 0 \\ 5/7 \end{bmatrix}$$

$$\begin{array}{c|ccc|c|l} & r & y_2 & r & 1 & \\ \hline & 0 & 3/4 & 1/4 & -1/4 & = -y_3 \\ & & & -1/10 & -1/10 & = -y_1 \\ \hline & -1/5 & -1/20 & -3/10 & 7/20 & = u \end{array}$$

$-\frac{9}{20}$ $\frac{1}{20}$

$-1/5 + \frac{1}{10}$

$1/5 + 3/20$

$\frac{3}{20} \times \frac{20}{7}$ $\frac{7}{20}$

$\frac{20}{7}$

Transportat° problem:

Example:

From \ to

	D_1	D_2	D_3	D_4	supply
W_1	22	36	24	23	10
W_2	31	19	32	26	40
W_3	25	25	16	22	50
Demand	20	30	30	20	

let x_{ij} $\begin{matrix} i=1,2,3 \\ j=1,2,3,4 \end{matrix}$ be the amount shipped from w_i to D_j

$$\text{minimize} \quad C = \sum_{i=1}^{3} \sum_{j=1}^{4} a_{ij}\, x_{ij}$$

$$\text{subject to} \quad \left.\begin{aligned} &\sum_{j=1}^{4} x_{1j} \leq 10 \\ &\sum_{j=1}^{4} x_{2j} \leq 40 \\ &\sum_{j=1}^{4} x_{3j} \leq 50 \end{aligned}\right\} \text{supply}$$

$$\left.\begin{aligned} &\sum_{i=1}^{3} x_{i1} \geq 20 \\ &\sum_{i=1}^{3} x_{i2} \geq 30 \\ &\sum_{i=1}^{3} x_{i3} \geq 30 \\ &\sum_{i=1}^{3} x_{i4} \geq 20 \end{aligned}\right\} \text{Demand.}$$

$$x_{ij} \geq 0$$

Assume supply = Demand.

Then the ~~eq~~ inequalities become equat^{ns} (=

# of basic variables = $m+n-1 = 6$ (in this case)

c_{ij} is called (i, j) cell

Terminology:

	H_1	H_2
W_1	2^{9}	3
W_2	4	5^{20}

(I) minimum entry method → to find an initial feasible pt.
(II) Stepping stone method → change the basic feasible point.
(III) U-V method → Determine the optimal solut^{n}.

Exple 1:

	D_1	D_2	D_3	D_4	supply
W_1	22^{10}	36	24	23_0	~~10~~
W_2	31^{10}	19^{30}	32	26	~~40~~ 10
W_3	25	25	16^{30}	22^{20}	~~50~~ 20
Demand	~~20~~ 10	30	30	~~20~~ 0	

delete the row bcz now the column are useless (∞)

Exple 2:

	5^{5}	10^{10}	5
	10^{5}	20	25
	~~10~~ 5	10	

we delete the great columns & the least r

~~th~~ $-2+t$

Exple 1 (continued):

22^{10}	36	24	23
31^{20-s}	19^{30}	32	26^{0+s}
$\boxed{25}^{s}$	25	16^{30}	22^{20-s}

31^{10} $\qquad$ 26^{0}

25 $\qquad$ 22^{20}

31^{10-s} $\qquad$ 20^{s}

25^{s} $\qquad$ 22^{20-s}

① $25s + 22\times(20-s) + 26s + 31\times(10-s)$ [new way]

② $25\times 0 + 22\times 20 + 26\times 0 + 31\times 10$ [old way]

① − ② $= -25 < 0 \Rightarrow$ the new way is the best way

the max value of s is 10 as $10-s \geq 0$, $s \geq 0$, $20-s \geq 0$ (the exponentials).

Correct of quiz:

- for eq^n of y_s choose rows whose coefs of x are $\neq 0$

for $i=1$: $4-5t$

for $i=2$: $-2+t$ $\Rightarrow 0$