MWF 11-11:50am - ARM 0131 sect 0251
Tu/Th 12-12:50pm - MTH 0306

$\boxed{\text{MATH 141}}$ Dr. R. Johnson
2107

Web Assign

http:// www. math. umd. edu/ users/ rlj/m141 f08. htm

On web Assign:

instead of	write
e^{2x}	exp(2x)
arctan	atan
∞	Infinity

Integrals:

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

$$\int_a^b f' = f + \underline{C}$$ (Web Assign wants the constant

Riemann Sum:

f on $[a,b]$

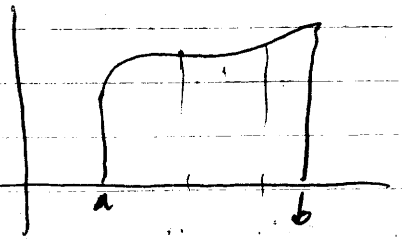
$P = \{a = x_0 < x_1 < \dots < x_n\}$

$t_i \in [x_{i-1}$

$$\int_a^b f(x)dx = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

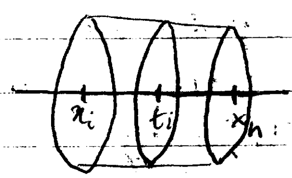
Regular partition $[a,b]$ is

$f(x) \geq 0$



$\int_a^b f$ = area under graph

$D = \{ a \leq x \leq b$, $A(x)$ cross-sect^e



$x_{i-1} \leq t_i \leq x_i$, $A(t_i)$

$A(x) \simeq A(t_i)$

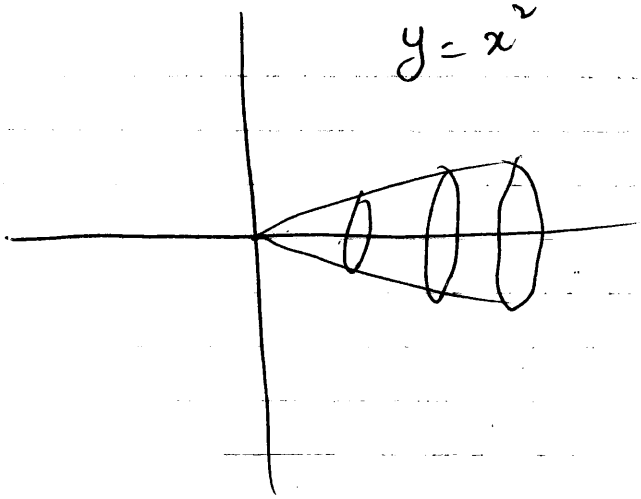
$\Delta V_i = A(x)\,(x_i - x_{i-1}) = A(t_i)(x_i - x_{i-1})$

Defin^l: If a volume lies on $a \leq x \leq b$ & at each x in cross-sect^{nal} area $A(x)$, then $V = \int_a^b A(x)\,dx$.
x

Cylendrical height h,

$A(x) = \pi r^2$, $a \leq x \leq$

$V = \int_a^{a+h} \pi r^2 \, dr = \pi r^2 h$



Cross sectional x
$0 \le x \le 1$ is a circle w/
radius x^2

$A(x) = \pi (x^2)^2$
$= \pi x^4$

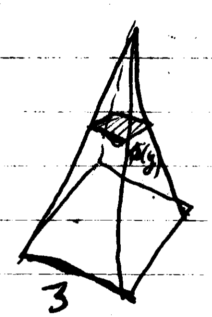
$$V = \int_0^1 \pi x^4 \, dx = \pi x^5/5 \Big|_0^1$$
$$= \frac{\pi}{5}$$

If cross-sectional area are known at each y.

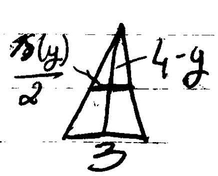
$$V = \int_c^d A(y)\, dy$$

Find the volume of a pyramid with a square base of length 3 and height 4.

$s(y) =$ side - length



$0 \le y \le 4$



$$\frac{s(y)}{4-y} = \frac{\frac{3}{2}}{2} \Rightarrow s(y) = \frac{3(4-y)}{2 \times 2} = 3\left(\frac{4-y}{4}\right)$$

$$A(y) = (s(y))^2$$

$$= \frac{3}{4}(4-y)^2$$

$$V = \frac{9}{16}\int_0^4 (4-y)^2\,dy$$

$$\frac{\frac{s(y)}{2}}{h-y} = \frac{b/2}{h} \longrightarrow \frac{s(y)}{b} = \frac{h-y}{h}$$

$$s(y) = b\left(\frac{h-y}{h}\right)$$

$$V = \int_0^h b^2 \frac{(h-y)^2}{h^2}\,dy$$

$= \frac{1}{3} b^2 h$

$$= \frac{b^2}{h^2}\int_0^h (h^2 - 2hy + y^2)\,dy$$

Circle radius $f(x)$

Even fct $f(-x) = f(x)$
then,
$\int_{-r}^{r} f = \int_0^r + \int_{-r}^0 = \int_0^r + \int_0^r$
$= 2\int_0^r f$

$$A(x) = \pi (f(x))^2$$

$$V = \int_a^b \pi f(x)^2\,dx$$

$$x^2 + y^2 = r^2$$

$$y = \sqrt{r^2 - x^2} \quad -r \le x \le r$$

uneven
$f(-x) = -f(x)$
then
$\int_{-r}^{r} f(x)dx = 0$

$$V = \int_{-r}^{r} A(x)\,dx$$

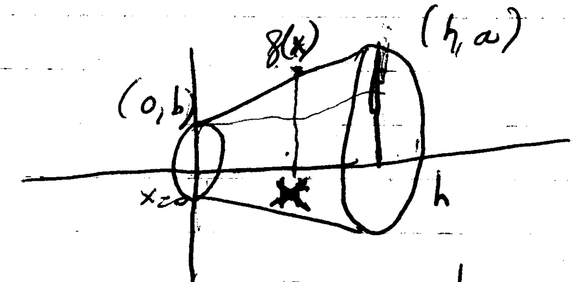
$$= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int^{r} (r^2 - x^2)\,dx = 2\pi \int^{r} (r^2 - x^2)\,d$$

$$= 2\pi \left(r^2 x - x^3/3\right)\Big|_0^r$$

$$= 2\pi \; \frac{2}{3} r^3 = \frac{4\pi r^3}{3}$$

Frustrum of a cone:



$0 \le x \le h$

b & a are radius of b

$$\frac{b - f(x)}{0 - x} = \frac{b - a}{0 - h}$$

$$\frac{f(x) - b}{x} = \frac{a - b}{h}$$

$$f(x) = b + \left(\frac{a-b}{h}\right)x$$

$$V = \pi \int_0^h \left(b + \frac{a-b}{h}x\right)^2 dx$$

$$u = b + \frac{a-b}{h}x$$

$$du = \frac{a-b}{h}\,dx$$

$$V = \pi \int_b^a u^2 \frac{h}{a-b}\,du$$

$$= \frac{\pi h}{a-b}\left(\frac{a^3 - b^3}{3}\right) = \frac{\pi h}{3}\left[a^3\left(\frac{a^3 - b^3}{a-b}\right)\right]$$

$V = \pi \int 9 - x^2$

$V = \pi \left[9x - \frac{1}{3}x^3 \right]\Big|_0^3$

$V = \pi \left[27 - \frac{1}{3}(27) - 0 \right]$

$V = \pi [27 - 9]$

$V = \pi [18]$

$V = \pi \int x^3 \sqrt{x^3 + 1}$

$=$

$\cancel{x^3 (x^3+1)^{-3}}$

$x (x^3+1)^{-3}$

$(g(f(x)))' = g'(f(x)) \cdot f'(x)$

$u (u^3+1)^{-3}$

$\cancel{3x} \quad \frac{1}{4} x^4$

$\int x^3 = \frac{1}{4} x^4$

$\int x = \frac{1}{2} x^2$

$f'(g(x)) \cdot g'(x)$

$f'(g(x)) = (x^3+1)$

$g'(x) = x$

$g(x) = \frac{1}{2} x^2$

$f(x) =$

$\frac{-1}{2} (x^3+1)^{-2}$

$\int x (x^3 + 1)^{-3}$

$f(g(x)) \cdot g'(x)$

$\frac{+1}{2} x^{+2} = g(x)$

$f'(g(x)) = (x^3+1)^{-3}$

http://www.webassign.net/und/login.html

19/05/2008.

math 141

until monday the 1rst homework

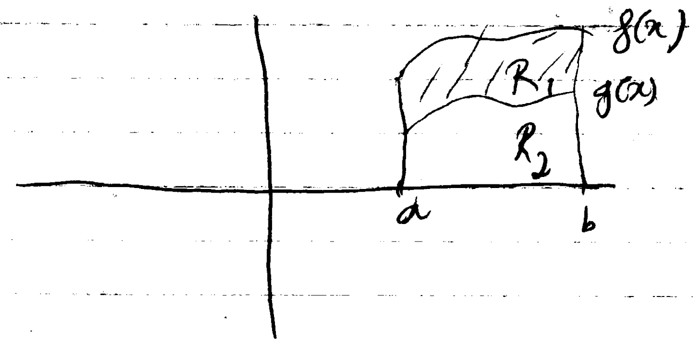
2) $V = \int A(x)\,dx$

Rotate region below graph of $f(x) \geq 0$ about x-axis

$$A(x) = \pi f(x)^2$$

$$V = \pi \int_a^b f(x)^2\,dx$$

$$R = \left\{(x,y) \quad \begin{matrix} a \leq x \leq b \\ 0 \leq g(x) \leq y \leq f(x) \end{matrix}\right.$$



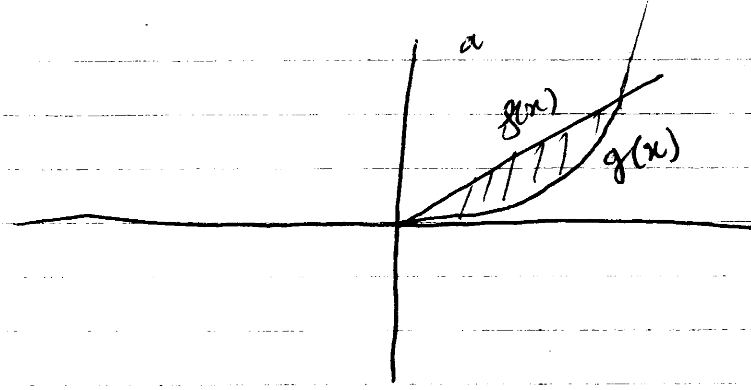
$R_1 = \{(x,y) \quad a \leq x \leq b, \quad 0 \leq y \leq f(x)\}$ $\quad V_{R_1} = \pi \int_a^b f(x)^2\,d$

$R_2 = \{(x,y) \mid a \leq x \leq b, \quad 0 \leq y \leq g(x)\}$

$$V_{R_2} = \pi \int_a^b g(x)^2\,dx$$

$$V_R = V_{R_1} - V_{R_2}$$

So $V = \pi \int_a^b \left[f(x)^2 - g(x)^2\right] dx$



$f(x) = 5x$

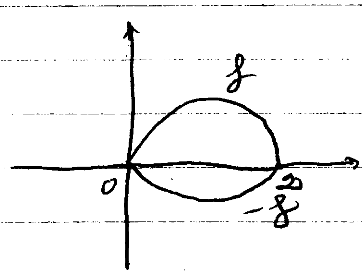
$g(x) = x^2 \quad 0 \le x \le 5 \qquad\qquad 0 \le x \le 5$

$V = \pi \int_0^5 [(5x)^2 - (x^2)^2]\, dx = \pi \int_0^5 (25x^2 - x^4)\, dx$

$R = \{ (x,y) \mid 0 \le x \le 10,\ \text{b/ graphs of } f \,\&\, g \}$

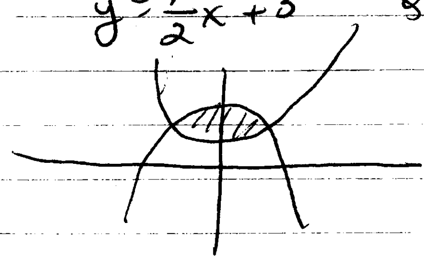
$V_1 = \pi \int_0^5 (25x^2 - x^4)\, dx + \pi \int_5^{10} [(x^2)^2 - (25x)^2]$

Ex: Find volume of reg^c b/ graphs of $f(x) = 2x - x^2$ & $g(x) =$ $g(x)$



$V = \pi \int_0^2 [2x - x^2]^2\, dx$

#15) $y = \frac{1}{2}x^2 + 3$ & $y = 12 - \frac{1}{2}x^2$



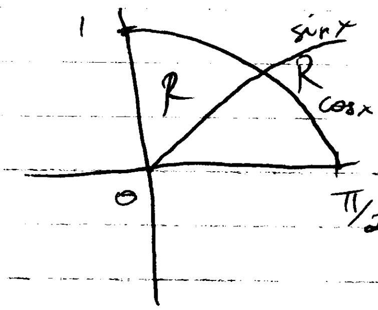
$\frac{1}{2}x^2 + 3 = 12 - \frac{1}{2}x^2$

$x^2 = 9 \qquad x = \pm 3$

so $V = \pi \int \left[(12 - \frac{1}{2}x^2)^2 - (\frac{1}{2}x^2 + 3)^2 \right] dx$

$$= \pi \int_{-3}^{3} 15(9 - x^2)\, dx$$

eg: Rotate reg^n b/ graphs of $\cos x$ & $\sin x$ on $[0, \frac{\pi}{2}]$



$\cos x = \sin x$
$x = \frac{\pi}{4}$

$$V = \pi \int_0^{\pi/4} [\cos^2 x - \sin^2 x]\, dx + \int_{\pi/4}^{\pi/2} [\sin^2 x - \cos^2 x]\, dx$$

$$= \pi \int_0^{\pi/4} \cos 2x\, dx + \int_{\pi/4}^{\pi/2} -\cos 2x\, dx = \pi \frac{1}{2} \sin 2x \Big|_0^{\pi/4} - \pi/2 \sin 2x \Big|_{\pi/4}$$

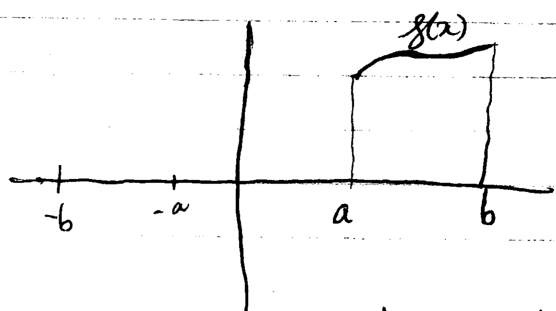
→ Graph $y = f(x)$

$R = \{(x,y) \mid a \le x \le b,\ 0 \le y \le f(x)\}$

$V = \pi \int_a^b (f(x)^2 - g(x)^2)\, dx$ (rotat^e about x-axis)

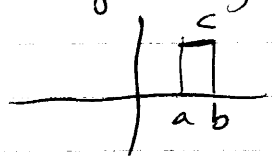
$R = \{(x,y) \mid c \le y \le d$
$g(x) \le x \le f(x)$

$V = \pi \int_c^d (f(y)^2 - g(y)^2)\, dy$ (rotat^e about y-axis)



rotate around the y axis

In case of a straight line



when rotating around y-axis, we get a cylind

So $V = \pi b^2 c - \pi a^2 c$

$V = \pi (b^2 - a^2) c$

In general,

let $P = \{ a \le x_0 \le x_1 \le \cdots \le x_n = b \}$

$V = \sum \Delta V_k \qquad \Delta V_k = \pi f(t_k)(x_k^2 - x_{k-1}^2)$

$$= \pi \sum_{k=1}^{n} f(t_k)(x_k^2 - x_{k-1}^2)$$

$$= 2\pi \sum_{k=1}^{n} f(t_k) \frac{(x_k + x_{k-1})}{2} (x_k$$

$$= 2\pi \sum_{k=1}^{n} (t f(t)) = \frac{x_k + x_{k-1}}{2} (x_k - x_{k-1}$$

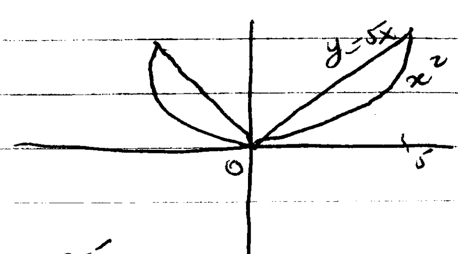
$$V = 2\pi \int_a^b x f(x)\, dx$$

$$R = \{ (x, y) \mid a \le x \le b \quad g(x) \le y \le f(x) \}$$

$$V = 2\pi \int_a^b x [f(x) - g(x)]\, dx$$

(rotating around the

e.g.:

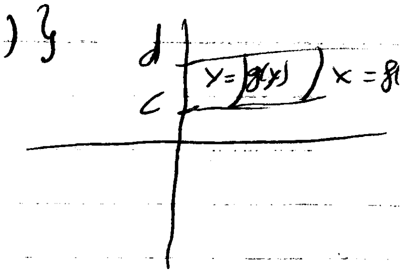


$0 \le x \le 5$

$$V = 2\pi \int_0^5 x [5x - x^2]\, dx$$

$$= 2\pi \int_0^5 (5x^2 - x^3)\, dx$$

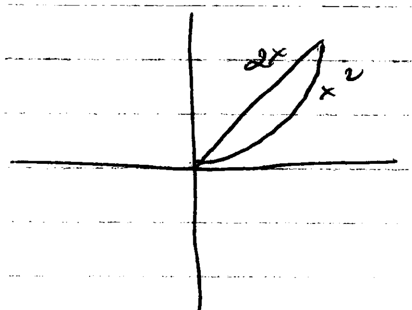
$R = \{(x,y) \mid c \le y \le d, \; g(y) \le x \le f(y)\}$



$V = 2\pi \int_a^b x(f(y) - g(y))\, dx$

$V = 2\pi \int_c^d y\,[f(y) - g(y)]\, dy$

e.g.: Region b/ graph of $y = 2x$ & $y = x^2$, $0 \le x \le 2$



rotate around x-axis

① $V = \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right] dx$

it doesn't matter what method U use (rotation from x or y-axis) U find the same answer

$= \pi \int_0^2 (4x^2 - x^4)\, dx$

$= \pi \left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2$

$= \pi \left(\frac{32}{3} - \right.$

$\boxed{V = \frac{64}{15}\pi}$

we want to rotate around the y-axis we find x

② Another w. $x = \frac{1}{2}y \quad x = \sqrt{y} \qquad 0 \le y \le 4$

$V = 2\pi \int_0^4 y\left[\sqrt{y} - \frac{1}{2}y\right] dy$

$= 2\pi \left[\frac{2}{5}y^{5/2} - \frac{y^3}{6}\right]\Big|_0^4$

$= 2\pi \left[\frac{4}{5} - \frac{64}{6}\right] = 2\pi\left(\frac{64}{}\right) = \boxed{64\pi}$

09/08/2008

4 methods:

→ Disc method

- rotate about the x-axis $\Rightarrow \pi \int_a^b f(x)^2$
  $y = f(x)$
- rotate about y-axis $\Rightarrow \pi \int_c^d g^2(y)$
  $x = g(y)$

→ washer method

- rotate about x-axis $\rightarrow \pi \int_a^b [f^2(x) - g^2(x)$
  $R = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le f(x)\}$
- – " – y-axis $\rightarrow \pi \int_c^d \{g^2(y) - h^2$

→ Shell method

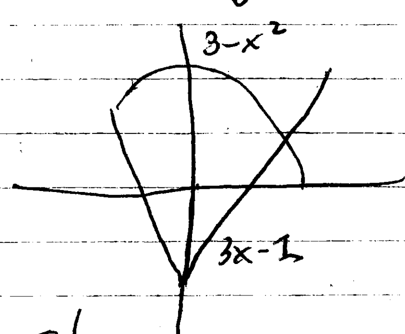
rotate reg^n b/ $y = f(x)$ & $y = g(x)$

about y-axis

$$V = 2\pi \int_a^b x\big(f(x) - g(x)\big)\,dx.$$

e.g.: Find V reg^n b/ graphs of $y = 3 - x^2$ & $y = 3x - 1$

(Shell method →) around y-axis



$3 - x^2 = 3x - 1$

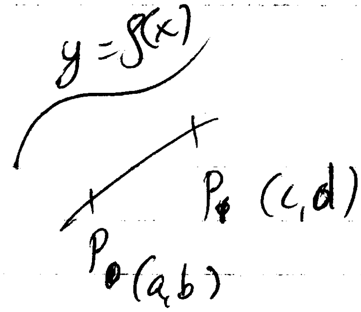
$0 = x^2 + 3x - 4$

$(x-1)(x+4)$

$$V = 2\pi \int_0^1 x\left[(3 - x^2) - (3x - 1)\right]dx = 2\pi \int_0^1 \left[x(4 - 3x - x^2)\right]dx$$

$$V = 2\pi \int_0^1 (4x - 3x^2 - x^3)\,dx$$

Length of a curve:



$$l = \sqrt{(c-a)^2 + (d-b)^2}$$

$$P = \{x_0 = a < x_1 < \dots < x_n = b\}$$

$$\Delta_k = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

$$l = \sum_{k=1}^{n} \Delta_k = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

f differentiable

$$f(x_k) - f(x_{k-1}) = f'(n_k)(x_k - x_{k-1})$$

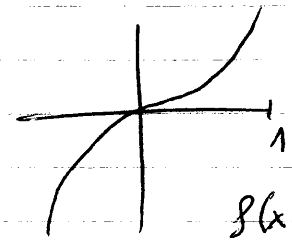
$$l = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + f'(n_k)^2 (x_k - x_{k-1})^2}$$

l = Riemann sum

$$= \sum_{k=1}^{n} \sqrt{1 + f'(n_k)^2}\, \underbrace{(x_k - x_{k-1})}_{\Delta x_k} = R\left(\sqrt{1+f'^2}, \{n_k\}, P\right)$$

$$\boxed{\text{Def } L = \int_a^b \sqrt{1 + f'(x)^2}\, dx}$$

e.g.: $y = \frac{1}{3}x^3 \;, \quad -1 \le x \le 1$



Sometimes it's impossible even for simple fct^e there is no integral

$$f(x) = \frac{1}{3}x^3$$

$$f'(x) = x^2$$

$$L = \int_{-1}^{1} \sqrt{1 + x^4}\, dx = 2\int_0^1 \sqrt{1 + x^4}\, dx$$ Cannot be eva

e.g.: $f(x) = 2x + 3 \quad 0 \le x \le 1$

$(0,3)$ to $(1,5)$ [straight line]

$$\sqrt{(5-3)^2 + (1-0)^2} = \underline{\underline{\sqrt{5}}}$$

$$\Updownarrow$$

$$f'(x) = 2$$

$$L = \int_0^1 \sqrt{1+4} = \underline{\underline{\sqrt{5}}}.$$

e.g.: $y = x^{3/2} + 1 \;, \quad 0 \le x \le 4$

$$f(x) = x^{3/2} + 1$$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$$

$u = 1+$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\; dx$$

$$u = 1 + \frac{9}{4}x$$

$$du = \frac{9}{4}dx$$

$$L = \int_0^4 u^{\frac{1}{2}} du \times \frac{4}{9}$$

$$L = \frac{4}{9}\int_0^4 u^{\frac{1}{2}} du$$

$$L = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]\Big|_0^4$$

$$L = \frac{4}{9}\left[\left(\frac{2}{3}(\sqrt{1+9})^{\frac{3}{2}} - \left(\frac{2}{3}(\sqrt{1+\right.\right.\right.$$

$$L = \frac{4}{9}\left(\frac{2}{3}(\sqrt{10})^{\frac{3}{2}} - \frac{2}{3}\right)$$

$$\boxed{L = \frac{8}{27}\left((\sqrt{10})^{3/2} - 1\right)}$$

$y = \ln x - \frac{x^2}{8}, \quad 1 \le x \le 2$

$f(x) = \ln x - \frac{x^2}{8}$

$f'(x) = \frac{1}{x} - \frac{1}{4}x$

$L = \int_1^2 \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\, dx$

$= \int_1^2 \sqrt{1 + \left(\frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}\right)}\, dx$

$= \int_1^2 \sqrt{\frac{1}{x^2} + 1 - \frac{1}{2} + \frac{x^2}{16}}\, dx$

$= \int_1^2 \sqrt{\frac{1}{2} + \frac{1}{x^2} + \frac{x^2}{16}}\, dx$

$= \int_1^2 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx$

$= \int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right) dx$

$= \int_1^2 \frac{1}{x}\,dx + \frac{x}{4}\, dx$

$= \ln x + \frac{x^2}{8} \Big|_1^2$

$\boxed{L = \ln 2 + \frac{3}{8}}$

$\frac{e^{2x} - 2e^{x}e^{-x} + e^{-2x}}{4}$

eg. #10

$f(x) = \frac{e^x + e^{-x}}{2} \quad 0 \le x \le \ln 2$

$f'(x) = \frac{e^x - e^{-x}}{2}$

$L = \int_0^{\ln 2} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx$

$L = \int_0^{\ln 2} \sqrt{1 + \frac{e^{2x}}{4} + \frac{e^{-2x}}{4} - e^{x}e^{-x}}\, dx$

$L = \int_0^{\ln 2} \sqrt{1 + \frac{1}{4}e^{2x} - \frac{2}{4} + \frac{e^{-2x}}{4}}$

(e^0)

$= \int_0^{\ln 2} \sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}\, dx$

$= \int_0^{\ln 2} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx$

$= \frac{1}{2}\int_0^{\ln 2} e^x + e^{-x}\, dx$

$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^{\ln 2}$

$= \frac{1}{2}\left[2 - \frac{1}{2} - (1 - 1)\right] =$

$\boxed{L = \frac{3}{4}}$

0500

Sect 6.4: Work

$W = F.d$ [Force $\times$ distance = Work]

Variable force $F(x)$ $a \le x \le b$

$P = \{ a = x_0 \le x_1 < \dots < x_n = b \}$

On $[x_{k-1}, x_k]$, $F(x) \simeq F(t_k)$ $x_{k-1} \le t_k \le x$

$\Delta W_k \simeq F(t_k)(x_k - x_{k-1})$

read the sect on units.

$W = \Sigma \Delta W_k = \Sigma$

Defin: The work done by a variable force $F(x)$, a
is $\boxed{W = \int_a^b F(x)\,dx}$

e.g.: Suppose a leaking wheelbarrow push 100 meter

$F(x) = 60\left(1 - \frac{x^2}{20{,}000}\right)$ $0 \le x \le 100$ / [the force we

$F(0) = 60$

$$W = \int_0^{100} 60\left(1 - \frac{x^2}{20{,}000}\right) dx$$

$$= \int_0^{100} \left(60 - \frac{60x^2}{20{,}000}\right) dx$$

$$= 60\left(x - \frac{x^3}{60{,}000}\right)\Big|_0^{100}$$

$=$

$$F = m\,a(x) = m\frac{dV}{dt} = m\frac{dV}{dx}\cdot\frac{dx}{dt} = m\frac{dV}{dx}V$$

$$\int F = \int \left(m v \frac{dv}{dx}\right) dx = \int m v\, dv = \frac{1}{2}m v_i^2 - \frac{1}{2}m v_0^2$$

The weight of ~~a~~ the water is 62.5 pounds per cubic foot
$\Rightarrow$ the weight of the water is $62.5V$

So $W = (\text{weight}) \times (\text{distance}) = 62.5Vd$
for the work done by the lifting force when pumping water out of a tank & up to a certain level.

If the liquid is another liquid $\neq$ water, 62.5 will be replaced by the weight of 1 cubic foot of the other liquid.

$$\boxed{W = \int_a^b 62.5\,(l - x)\,A(x)\,dx}$$

Gravitational force $F = -\frac{GMm}{r^2}$

$$W(b) = \int_R^b -\frac{GMm}{r^2}\,dr = \frac{GMm}{r}\Big|_R^b$$

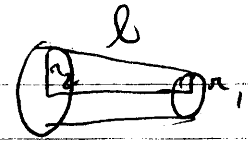
= radius of the earth.

$$W(b) = GMm\left(\frac{1}{b} - \frac{1}{R}\right) = \left(\frac{1}{2}m(V(b))^2 - \frac{1}{2}m[V(R$$

(Work done when spaceship travels a distance b from the ~~center of~~ earth surface)

09/10/2008

$A_{cylinder} = 2\pi a h$

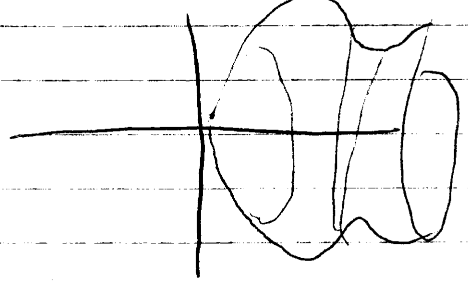


$A = 2\pi\left(\frac{r_1+r_2}{2}\right) l$

$A = \pi(r_1+r_2)\, l$

$L = \int_a^b \sqrt{1+f'(x)^2}\,dx$

$y = f(x)$ rotate around x-axis



$P = \{a = x_0 \le x_1 < \cdots < x_n = b\}$

on $[x_{k-1}, x_{k+0}]$

$\cancel{\Delta S_k = \pi\,[f(x_{k-1}) + f(x_k)]\, f(x_{...}}$

$\Delta S_k = \pi\,[f(x_{k-1}) + f(x_k)]\sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f ...}$

S = Surface Area.

$= \pi\,[f(x_{k-1}) + f(x_k)]\sqrt{1+f'(t_k)^2}\,(x_k - ...$

$= \pi\, 2 f(t_k)\sqrt{1+f'(t_k)^2}\,\Delta x_k$

Def. $S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$.

e.g.: $y = x^3$ on $[0,1]$

$S = 2\pi\int_a^b x^3\sqrt{1+(3x^2)^2}\,dx$

$= 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx = 2\pi\int_0^1 u^{\frac{1}{2}}\frac{du}{36}$

let $u = 1+9x^4$

$du = 36x^3$

$= \frac{\pi}{18}\left[\left(\frac{2}{3}u^{3/2}\right)\right]\Big|_0^1$

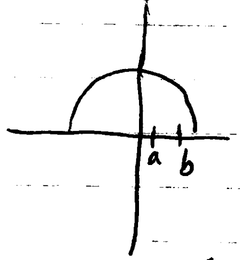
$= \frac{\pi}{18}\left(\frac{2}{3}\sqrt{(1+9x^4)^3}\right)$

$= \frac{\pi}{\ldots}\left[10^{3/2} - 1\right]$

e.g.: $x^2+y^2=1$ (Cercle)

$$y^2 = 1-x^2$$

$$y = \sqrt{1-x^2}$$



$$-1 \le a \le b \le 1$$

$$f(x) = (1-x^2)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

$$1+(f'(x))^2 = 1+\frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$S = 2\pi\int_a^b (1-x^2)^{1/2}\,\frac{1}{\sqrt{1-x^2}}\,dx$$

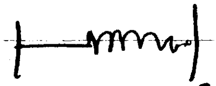
$$= 2\pi(b-a)$$

Sphere of radius 1 has surface area 4π

$$= 2\pi[1-(-1)]$$

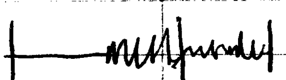
Hook's law: Restoring force exerted by a strong ~~steering~~ stretch a x units from equilibrium is

K = spring constant



$F(x) = kx$

e.g.: Work required to stretch a spring 1 unit from equilibrium is 10^6 ergs. Find the amount of work required to stretch to 3 units from equilibrium.



$x=0$ 1

$$W = \int_0^1 kx\,dx = 10^6 \qquad 10^6 = \int_0^1 kx\,dx$$

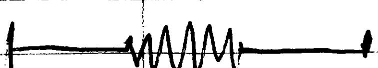
$$W_1 = \int_1^3 kx\,dx \qquad k = 2\times10^6$$

$$= 2\times10^6 \int_1^3 x\,dx$$

$$= 2\times10^6 \left(\tfrac{1}{2}x^2\right)\Big|_1^3 = 8\times10^6$$

e.g.: ~~lxo~~ 8 10 cm from equilibrium ~~10 cm~~ $F = 4\times10^6$ dynes

Additional 10 cm → W?



10 cm

$$F = kx \Rightarrow k = 4\times105$$

$$W = \int_{10}^{20} (4\times105)\, x\,dx$$

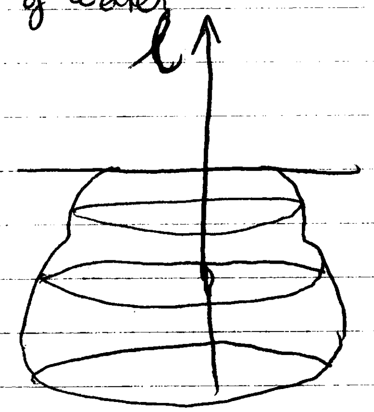
Work done in moving volume of

Density of water $= 62.5\ lb/ft^3$

Move a volume of water V a distance d.

$$W = Fd = (62.5V)\,d.$$

pool of water



ℓ

$x=0$

$$a \le x \le b \le 0$$

$$P = \{a = x_0 \le x_1 < x_2 < \cdots < x_n = b\}$$

$$[x_{k-1}, x_k] \quad x \in [x_{k-1}, x_k]$$

$$d = \ell - t_k\ , \quad x_{k-1} \le t_k \le$$

$$\Delta V_k = A(t_k)(x_k - x_{k-1})$$

x_k

x_{k-1}

$$\Delta V_k = \int_{x_{k-1}}^{x_k} A(x)\,dx \simeq A(t_k)($$

M T W Th F

$$\Delta W_k = 62.5\, A(t_k)(l - t_k)(x_k - x_{k-1})$$

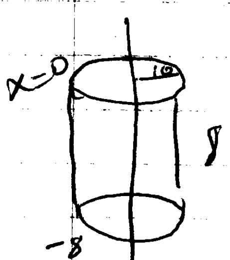
$$W = \sum_{k=1}^{n} \Delta W_k$$

$$= 62.5 \sum_{k=1}^{n} R\left(A(x)(l-x),\ P,\ \{t_k\}\right)$$

$$\downarrow$$

Def. $W = 62.5 \int_a^b A(x)(l-x)\,dx$.

e.g.



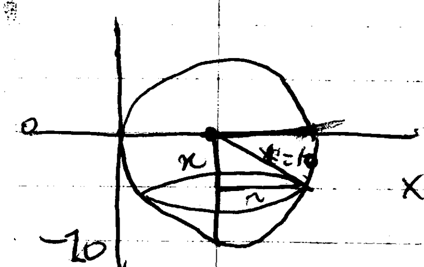
$l = 0$

we want to pull all the water out

$$W = 62.5 \int_{-8}^{0} (0-x)(100\pi)$$

$$A(x) = \pi(10)^2 = 100\pi$$

e.g. the pool is hemispherical



$$x^2 + r^2 = 10^2$$

$$r(x^2) = 100 - x^2$$

$$W = \int_{-10}^{0} 62.5(0-x)\pi(100 - x^2)\,dx$$

$$= 62.5 \int_{-10}^{8}$$

(l is defined as the height of the ~~solid~~ water in the ~~solid~~)

$$W = \int F(x)\,dx$$

$$F(x) = m\,a(x) = m\frac{d^2x}{dt} = m\frac{dv}{dt}$$

$$= m\frac{dv}{dx} \times \frac{dx}{dt}$$

$$= m \times \frac{dv}{dx} \times v$$

$$W = \int_a^b F(x)\,dx$$

$$= \int_a^b mv\frac{dv}{dx}$$

$$= \frac{1}{2}mv^2 \Big|_a^b$$

$$= \frac{1}{2}mv(b)^2 - \frac{1}{2}mv(a)^2$$

$$F(r) = G\frac{Mm}{r^2}$$

$$W = \int_R^b F(r)\,dr = \int -\frac{GMm}{r^2}\,dr = +GMm\,\frac{1}{r}\Big|_R^b$$

$$= GM_m\left(\frac{1}{b} - \frac{1}{R}\right)$$

$$\frac{1}{2}mV(b)^2 - \frac{1}{2}mV(R)^2 = GMm\left(\frac{1}{b} - \frac{1}{R}\right)$$

$$\frac{1}{2}mV(b)^2 - GMm/b = \frac{1}{2}mV(R) - \frac{GMm}{R}$$

What is the escape velocity?

$$\lim_{b\to\infty} V(b) = 0$$

$$\frac{1}{b} \to 0$$

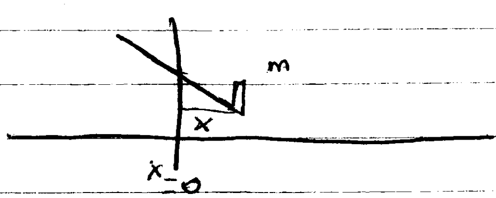
$$LHS \to 0$$

$$0 = \frac{1}{2}mV(R^2 - \frac{GMm}{R^2}$$

$$V(R)^2 = \frac{2GM}{R^2}$$

$$V(R) = \sqrt{2GM}/R.$$

Moment & center of mass:



~ weight (m_1, x_1)

$$M_y = mx_1$$
$$M_y = m_1x_1 + m_2x_2 + \dots + m_nx_n = m\bar{x}$$

Center of mass $(\bar{X}, \bar{Y})$

$$\bar{X} = \frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m_1 + m_2 + \dots + m_n}$$

$$m = m_1 + m_2 + \dots + m_n$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + \dots + m_ny_n}{m_1 + m_2 + \dots + m_n}.$$

e.g. $m_1 = 2(1,2)$
$m_2 = 4(-2,1)$
$m_3 = 5(-1,-2)$
$m_4 = 7(0,3)$

$$m = 7+5+4+2 = 18$$

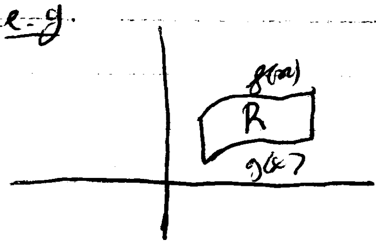
$$M_y = 2\cdot 1 + 4(-3) + 5(-1) + 7(0)$$
$$= -15$$

$$M_x = 2\cdot 2 + 4\cdot 1 + 5(-2) + 7(3) = 19$$

$$\bar{x} = -5/6 = \frac{M_y}{18}$$

$$\bar{Y} = 19/18 \cdot M_x$$

e.g.



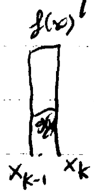
$y = f(x) \quad 0 \le g(x) \le f(x)$

$y = g(x) \quad a \le x \le b$

$m \approx$ area of R

$M_y = \sum m_i x_i$

$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$



$\Delta A_k = [f(t_k) - g(t_k)](x_{k-1} - \cancel{x_k})$

$m_k = \Delta A_k = [f(t_k) - g(t_k)](x_k - x_{k-1})$

$M_y/A \quad M_x/A$

$x_k = t_k,$

$x_{k-1} \le t_k \le x_k$

$r\ f(x) > g(x) \Rightarrow$

Def: $\boxed{M_y = \int_a^b x[f(x) - g(x)]\,dx}$

$\boxed{M_y = \int_a^b x\, f(x)\,dx}$ (for only 1 fct)

e.g.

$f(x) = c$ on $[a,b]$

$M_y = \int_a^b cx\,dx = c\frac{1}{2}x^2\Big|_a^b$

$= c\left[\frac{1}{2}b^2 - \frac{1}{2}a^2\right]$

$= \frac{1}{2}c(b-a)(b+a)$

$= c(b-a)\,\frac{1}{2}(b+a)$

$=$

Def: $\boxed{M_x = \frac{1}{2}\int_a^b f^2(x)\,dx}$

mass R = area R

$M = \int_a^b (f(x) - g(x))\,dx$

$\bar{x} M = \int_a^b x(f(x) - g(x))\,dx$

$\boxed{\bar{x} = \dfrac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b (f(x) - g(x))\,dx}}$

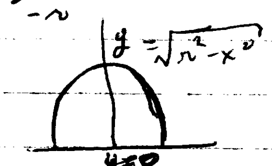
$M\bar{y} = \frac{1}{2}\int_a^b (f^2(x) - g^2(x))\,dx$

$\bar{y} = \dfrac{\frac{1}{2}\int_a^b [f(x)^2 - g(x)^2]\,dx}{\int_a^b (f(x) - g(x))\,dx}$

$x\sqrt{r^2 - x^2}$ is odd $\Rightarrow f(x) = -f(-x)$

$\Rightarrow 2\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx = 0$

e.g.



$M_y = \int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx = 0$

$M_x = \frac{1}{2}\int_{-r}^{r} [(r^2 - x^2) - 0^2]\,dx$

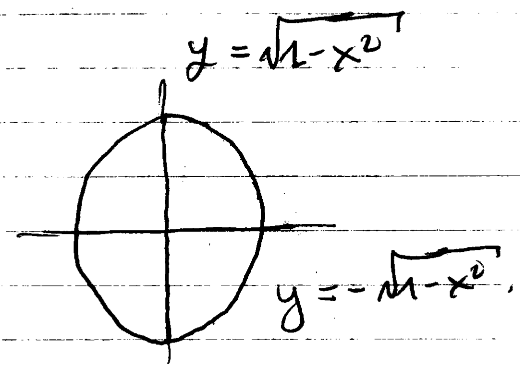
$= \int_0^r (r^2 - x^2)\,dx = \frac{2}{3}r^3$

Fact. • If a reg^e R is symetric about $x=c$, its center of ~~mass~~ satisfies $\bar{x}=c$.

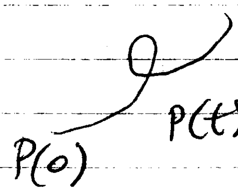
• If R is symetric about $y=d$, $\bar{y}=d$.

Read
Pappos - Guldin

Parametized curves



$y=f(x)$



$P(t)=(x(t), y(t))$

$a \le t \le b$

$P(t)=(x(t), y(t))$

$\{P(t) \mid t \in [a,b]\}$

e.g. Point moves on parameterized curve

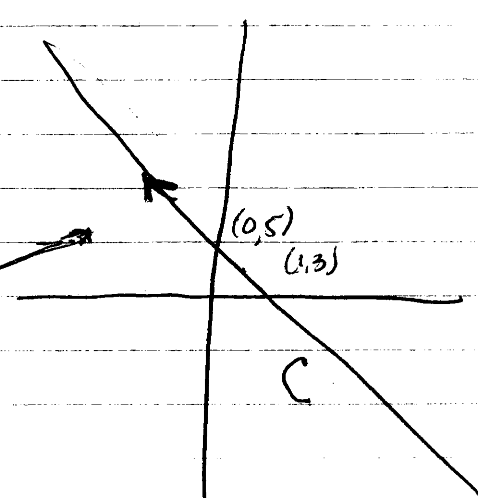
$x(t)=1-2t$
$y(t)=3+4t$

$2t=1-x$
$t=\frac{1-x}{2}$

$y=3+4\left(\frac{1-x}{2}\right)$

$\therefore$ ~~ko~~ $y=5-2x$

$t=0 \Rightarrow x=1$ & $y=3$
$t=1 \rightarrow P(1)=(-1,7)$



Any straight line is parametrized by $x = a + bt$ $\quad bt = x - a$

$y = c + dt$ $\quad t = \frac{x-a}{b}$ $\quad b.$

If $b = 0$, $x = 0$

$y = c + dt$

e.g.: find parametric line that passes through $(-2,3)$ & $(2,0)$

$x = a + bt \qquad P(0) = (-2,3) = (a, c)$

$y = c + dt \qquad P(1) = (2,0) = (a+b, c+d)$ (replace t=

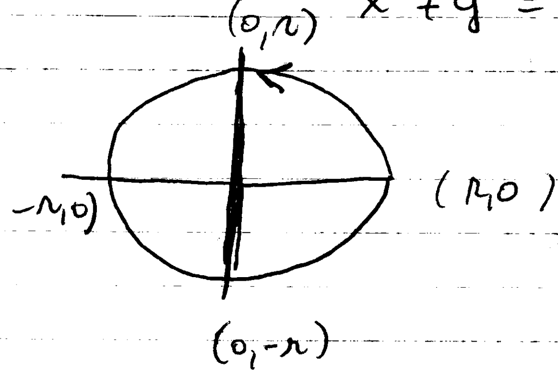
$P(1) =$

$x = -2 + 4t, \quad y = 3 - 3t.$

e.g.: $x = r\cos t \qquad x/r = \cos t$

$y = r\sin t \qquad y/r = \sin t$

$x^2 + y^2 = r^2 \qquad 0 \le t \le 2\pi.$



As $t\uparrow$ we move from (0,1) to $(0,r)$ clockwise.

Angular velocity:

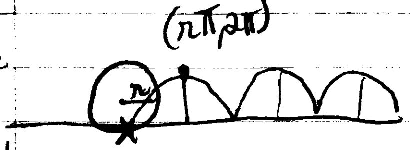
$\frac{d\theta}{dt} \qquad \theta(t) = \omega t \qquad \frac{d\theta}{dt} = \omega$

$x = r\cos\omega t \qquad 0 \le t \le \frac{2\pi}{\omega}$

$y = r\sin\omega t$

Cycloid

When the pt from the circle rolls on the line (road) the pt forms a cycloid



$$x(t) = r(t - \sin t)$$
$$y(t) = r(1 - \cos t)$$

$P(0) = (0,0)$ $\qquad \cos t = -1$
$\qquad t = \pi + 2r\pi$

Lenght of a parametrized curve:

$$P(t) = (x(t), y(t)), \quad a \le t \le b.$$

$$P = \{a = t_0 < t_1 < \dots < t_n = b\}$$

$$P_k = (x(t_k), y(t_k))$$

$$\Delta L_k = \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}$$

$$L = \sum_{k=1}^{n} \Delta L_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}$$

$$x(t_k) - x(t_{k-1}) = x'(V_k)(t_k - t_{k-1})$$
$$y(t_k) - y(t_{k-1}) = y'(V_k)(t_k - t_{k-1})$$

$$L = \sum_{k=1}^{n} \sqrt{x(t_k)^2 + y(V_k)^2}\,(t_k - t_{k-1})$$

not a $\downarrow$ Riemann sum

Def $\qquad L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$\cos 2t = 1 - 2\sin^2 t.$

$\cos t = 1 - 2\sin^2 t/2$

$\boxed{2\sin^2 t/2 = 1 - \cos t}$

$f(g(x)) = g'($

$x = r(t - \sin t) = rt - r\sin$

$y = f(x) \quad a \le x \le b$

$x = t, \quad y(t) = f(t) \qquad a \le t \le b.$

$x' = 1 \qquad y' = f'(t)$

$$L = \int_a^b \sqrt{1^2 + f'(t)^2}\, dt.$$

e.g.: $x(t) = r\cos t$

$y(t) = r\sin t \qquad 0 \le t \le 2\pi$

$x' = -r\sin t$

$y' = r\cos t$

$$L = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$$L = \int_0^{2\pi} r\, dt = 2\pi r = \pi d. \qquad = \pi d.$$

e.g. $x = r(t - \sin t)$

$y = r(1 - \cos t)$

Calculate length of an arch cycloid

$x' = r(1 - \cos t) \qquad 0 \le t \le 2\pi$

$y' = r\sin t.$

$$L = \int_0^{2\pi} \sqrt{r^2(1-\cos t)^2 + r^2\sin^2 t}\, dt$$

$$= r\int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos t)}\, dt$$

$$= r\int_0^{2\pi} \sqrt{4\sin^2 t/2}\, dt$$

$$= 2r\int_0^{2\pi} |\sin t/2|\, dt$$

$$= 2r\left[2\cos t/2\right]_0^{2\pi} = 2r[2+2] = 8r.$$

e.g.: ℓ

t parameter $a \le t \le b$ $(x(t), y(t))$

T time ~~allowing~~ along curve $P(t)$

$T = 0$

L = length along curve from 0 to t.

Speed $V = \dfrac{dL}{dT}$ Speed alond the parameterized cu[rve] $P(t)$

$$\frac{dL}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

T. circle with angular velocity w.

$\theta(T) = \omega T$ $\qquad T(t) = \omega T$

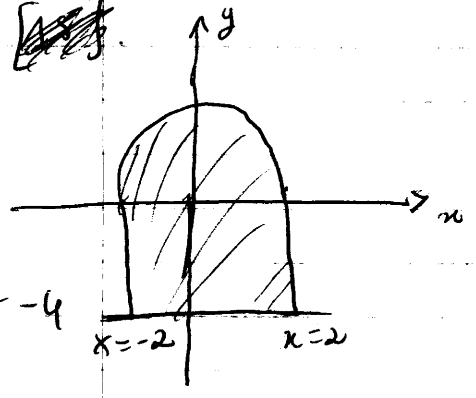
$x(T) = r\cos\omega T$

$y(T) = r\sin\omega T$

$$V = \frac{dL}{dT} = \sqrt{x'(T)^2 + y'(T)^2}$$

$$= \sqrt{r^2\omega^2\sin\omega T + r^2\omega^2\cos^2\omega T}.$$

source



$y = \sqrt{4-x^2}$

$$M_x = \int_a^b \frac{1}{2}\left(f^2(x) - g^2(x)\right) dx$$

$$M_y = \int_a^b x\left(f(x) - g(x)\right) dx$$

$$A = \int_a^b \left(f(x) - g(x)\right) dx$$

$f(x) = \sqrt{4-x^2}$

$g(x) = -4.$ for $x \in (-2, 2)$

$$M_x = \int_{-2}^{2} \frac{1}{2}(4 - x^2 - 16)\, dx = \frac{-80}{3}$$

$$\bar{y} = \frac{M_x}{A} =$$

gave it to me

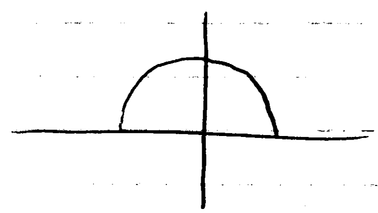
$1/x$ $\ln x + \ln y$ $-1/x$ $+ \frac{1}{y}\frac{dy}{dx}$

$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$ for a surface created by revolving $y = f(x)$ about x-axis

Parametric curve $x = f(t)$, $y = g(t)$

$$S = 2\pi \int_a^b g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Surface area of a sphere of radius



$x = r\cos t$, $y = r\sin t$ $0 \le t \le$

$$S(r) = 2\pi \int_0^{2\pi} r\sin(t)\sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$$= 2\pi r^2 \int_0^{\pi} \sin t\, dt$$

$$= 2\pi r^2 \left(-\cos t \Big|_0^{\pi}\right)$$

$$= 2(2\pi r^2) =$$

$$\boxed{S = 4\pi r^2}$$

Formulas: Volume / Area / Length

One arched cycloid

$x = r(t - \sin t)$ $\quad 0 \leq t \leq 2\pi$

$y = r(1 - \cos t)$

$f(a) = c$

$f^{-1}(c) = a$

$(f^{-1})'(c) = \frac{1}{f'(a)}$

$f(t) = r(t - \sin t)$

$g(t) = r(1 - \cos t)$

$f'(t) = r(1 - \cos t)$

$g'(t) = r \sin t$

$S = 2\pi \int_0^{2\pi} r(1-\cos t)\sqrt{r^2(1-\cos t^2) + r^2 \sin^2 t}\, dt$

$2 - 2\cos t = 2(1 - \cos t) = 4\sin^2 t/2$

$S = 2\pi r^2 \int_0^{2\pi} (1 - \cos t)\, 2\, |\sin t/2|\, dt$

chp ~~7~~ 7

8

09/21/08

Inverse of a fct:

$\boxed{g = f^{-1}}$

f has an inverse if there is a fct g with ~~domain~~ domain g = range f & range g = domain f, such that $f(x) = y$ if & only if $g(y) = x = f^{-1}(x)$.

reciprocal: $1/f$

1) Find the inverse fct (calculate)
2) When does it exist.
3) Continuity & rule for differentiability on inverse.

$x^3 = y$

solve for x as a fct of y

$x = y^{\frac{1}{3}}$

so $f^{-1}(x) = x^{1/3}$

$f^{-1}(8) = 8^{1/3} = 2$

$1/f(8) = \frac{1}{f(8)} = \frac{1}{8^3} = \frac{1}{512}$

1) **Rule:** to find inverse fcts:
$f(x) = y$
- solve for x as a fct of y
- the solut is the inverse fct: $x = g(y)$.

e.g.: $f(x) = 8x^3 - 1 = y$

$8x^3 = y + 1$

$x^3 = \frac{y+1}{8}$

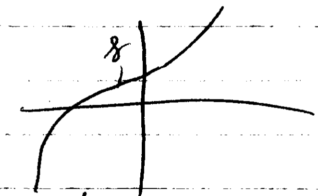
$x = \sqrt[3]{\frac{y+1}{8}}$

$f^{-1}(y) = \left(\frac{y+1}{8}\right)^{1/3}$

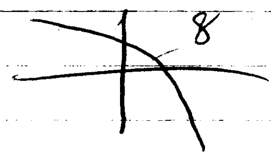
$\boxed{f^{-1}(x) = \left(\frac{x+1}{8}\right)^{\frac{1}{3}}}$

2) When does f has an inverse?

f is strictly increasing if $x < y$ implies $f(x) < f(y)$



f is strictly decreasing if $x < y$ implies $f(x) >$



$f^{-1} = g$ is a fct

Suppose f has an inverse

$f(x_1) = f(x_2)$

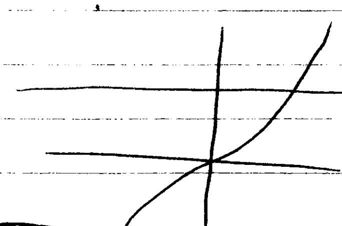
f has an inverse if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (one to one fct)

$g(x_1) = x_1$

$g(x_2) = x_2 \implies x_1 = x_2$

fct is one to one when $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Horizontal line test

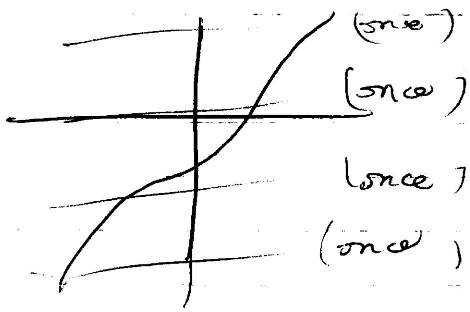


if the horizontal line cr the graph once $\Rightarrow$ one to o $\Rightarrow$ fx has ~~to~~ an inver

To see if f has an inverse
- Check this condit° $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- Apply the horizontal line test

e.g.

$f(x) = 8x^3 - 1 = y$



$\Rightarrow$ f has an inverse.

- If f is <u>strictly increasing</u>, it is one-to-one, & thus f has an inverse.

- If $f'(x) > 0$ on I, $\forall x$, except on a finite # of pts. (or $f'(x) < 0$), f is strictly increasing on I & thus has an inverse

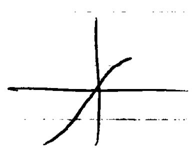
e.g. $f(x) = x^2 \quad x \geq 0$

$f'(x) = 2x > 0$ if $x > 0$ (& $= 0$ if $x = 0$).

$g(x) = x^3$

$g'(x) = 3x^2 > 0$ if $x \neq 0$

e.g. $y = \sin x$ on $[-\frac{\pi}{2}, \pi/2]$



$f'(x) = \cos x > 0$ on $[-\pi/2, \pi/2]$ except on $\pi/2$ & $-\pi/2$.

So $f(x)$ is strictly increasing

e.g. $f(x) = x^7 + 35x^5 + 14x^3 + 4x$ ← (impossible to find inverse)

$f'(x) = 7x^6 + 175x^4 + 42x^2 + 4$

$f'(x) \geq 4$ for all x.

Therefore f has an inverse

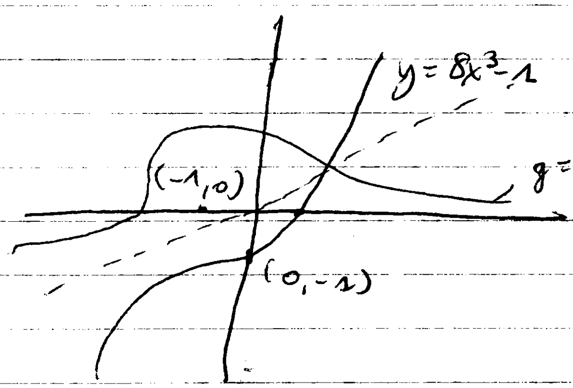
Graph of inverse fct^e.

(a,b) is on the graph of f means $f(a) = b$.

$g(b) = a$

⇓

(b,a) is on the graph of $g = f^{-1}$



g is the mirror image of the graph of f by mirror $y = x$.

3) If the fct^e is continuous & has an inverse, what can say about the inverse?

Theorem: If $f: I \to J$ in I is continuous & has an inverse $f^{-1}: J \to I$ is continuous.

If f is differentiable at a pt (a) & $f(a) \neq 0$, f^{-1} will be differentiable (but not at a) at $c = \cancel{f^{-1}(a}$

$c = f(a)$ & $(f^{-1})'(a) = \dfrac{1}{f(a)} = \dfrac{1}{f'(f^{-1}(a))}$

e.g: $y = 8x^3 - 1$

$f(1) = 7 \quad f^{-1}(7) = 1 \qquad f'(1) = 24x^2 \quad f'(1) = 24$

$$(f^{-1})'(7) = \frac{1}{f'(1)} = \frac{1}{24}$$

$$f^{-1}(x) = \left(\frac{x+1}{8}\right)^{1/3}$$

$$(f^{-1})'(x) = \frac{1}{3}\left(\frac{x+1}{8}\right)^{(-2/3)}\left(\frac{1}{8}\right) = \frac{1}{24}\left(\frac{x+1}{8}\right)^{-2/3}$$

$$(f^{-1})'(7) = \frac{1}{24}$$

$$(f^{-1})'(c) = \lim_{y \to c} \frac{f^{-1}(y) - f^{-1}(c)}{y - c}$$

$c = f(a)$

$f^{-1}(c) = a$

$$= \lim_{y \to c} \frac{x - a}{f(x) - f(a)}$$

$x = f^{-1}(y)$

$y = f(x)$

$$= \lim_{y \to c} \frac{1}{\frac{f(x) - f(a)}{x - a}}$$

$$(f^{-1})'(c) = \lim_{y \to c} \frac{1}{f'(a)}$$

Since f is derivable at a, then f is continuous at a,

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{y \to c} f^{-1}(y) = f^{-1}(a)$$

$$\lim_{y \to c} x = a$$

(the derivative of the inverse is the reciprocal of the derivative ~~of~~)

$\cancel{y = f(x)} \qquad \cancel{y = x = f^{-1}(y)}$

$\frac{d(x)}{dy} = \frac{1}{dy/dx(x)}$ $\qquad \frac{dy}{dx} \qquad \cancel{\frac{dx}{dy}} =$

$f(x) = \int_0^{x^3} \sin^6(t^2)dt = G(x^3), \qquad G(x) = \int_0^x \sin^6(t^2)_G$

Does f have an inverse? $\qquad G'(x) = \sin^6(x^2)$

$f(x) = G(x^3)$

$f'(x) = G'(x^3) \cdot 3x^2 =$

$= 3x^2 \sin^6((x^3)^2)$

$= 3x^2 \sin^6(x^6) \geq 0$

$\sin^6 x^6 = \cancel{x^6} = n\pi$ on $[a,b]$

$f'(x) > 0$ except a finite # of pts.

$\Rightarrow f$ is strictly increasing on $[a,b]$

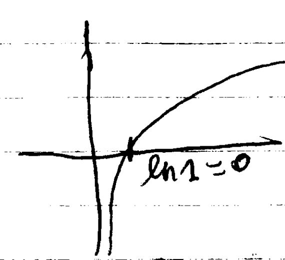
$c = f(\sqrt[6]{\frac{\pi}{6}})$ what is $(f^{-1})'(c) = \frac{1}{f'(\sqrt[6]{\pi/6})}$

$f'(x) = 3x^2 \sin^6(x^6)$

$f'(\sqrt[6]{\pi/6}) = 3\left(\frac{\pi}{6}\right)^{1/3} \sin^6\left(\frac{\pi}{6}\right) = \frac{3}{64}\sqrt[3]{\pi/6}$.

$\ln x = \int_1^x \frac{1}{t}\, dt \quad , x > 0$

$\frac{d}{dx} \ln x = \frac{1}{x} > 0$



$$\ln : x;\ \overset{D}{(0,+\infty)} \to \overset{\text{Range}}{(-\infty,\infty)}$$

$$e^x : x;\ (-\infty,\infty) \to (0,+\infty)$$

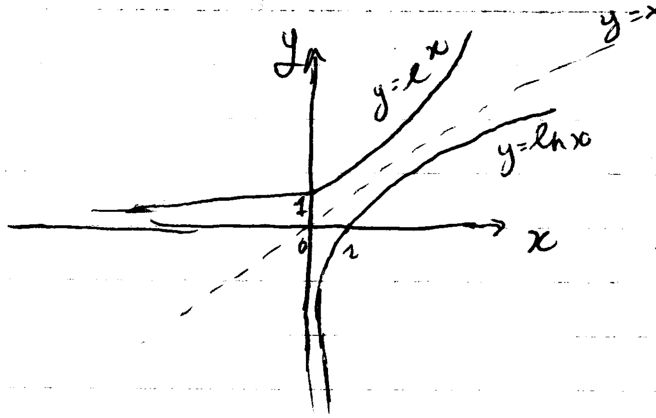
$\ln 1 = 0$

$e^0 = 1$

$e^{\ln x} = x,\ x > 0$

$\ln e^x = x,\ -\infty < x < \infty$

Definit^{n}: e^x is the inverse fct^{e} of $\ln x$.



$\lim\limits_{x \to -\infty} e^x = 0$

$\lim\limits_{x \to \infty} e^x = \infty$

$\lim\limits_{x \to 0^+} \ln x = \infty$

$\lim\limits_{x \to \infty} \ln x = +\infty$

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

Def.: e^x is the inverse of $\ln x$

$y = e^x \quad x = \ln y$

$$\frac{dy}{dx} = \frac{1}{\frac{d}{dy}\ln y}$$

$$= \frac{1}{\frac{1}{y}}$$

$$= y$$

$$\frac{dy}{dx} = e^x \qquad -\infty < x < +\infty.$$

$\Rightarrow e^x$ is strictly increasing

$$\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)$$

$$\frac{d}{dx} e^{g(x)} = g'(x) \cdot e^{g(x)}$$

$$\int g'(x) e^{g(x)} dx = e^{g(x)} + C$$

e.g.

$$\int e^{k(x)} dx = \int e^u \frac{du}{k}$$

$u = kx$

$du = k\, dx$

so

$$\int e^{k(x)} dx = \frac{1}{k} \int e^u du$$

$$= \frac{1}{k} e^u + C$$

$$\boxed{\int e^{k(x)} dx = \frac{1}{k} e^{kx} + C}$$

$\int e^{-x^2}dx$

can't be evaluated in closed form.

<u>Fact</u>: $e^{b+c} = e^b e^c$, b, c real

$\ln(xy) = \ln x + \ln y$
$\ln x^r = r \ln x$

$\ln e^{b+c} = b+c$ by definite of the inverse
$\ln e^b e^c = \ln(e^b) + \ln(e^c)$
$= b+c.$

$\ln(e^{b+c}) = \ln(e^b e^c)$
$\ln$ is $1:1$ because it's increasing).

$\Downarrow$

$e^{b+c} = e^b e^c$ for b, c real

$c = -b,\ b+c=0$
$1 = e^b e^{-b} \qquad e^{-b} = 1/e^b$

$e^{b-c} = e^{b+(-c)}$
$= e^b e^{-c}$
$= e^b / e^c$

$\int e^{\sin x} \cos x\, dx = ? \quad = \int e^u\, du = e^u + C = e^{\sin x} + C.$
$u = \sin x$
$du = \cos x\, dx$

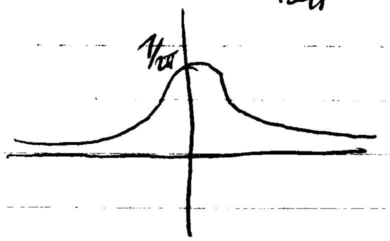
$f(x) = e^x - \ln x$, $x > 0$
if $0 < x < 1$, $\ln x < 0$, $e^x > 0$
$x \geq 1$, $f(1) = e - \ln 1 = e > 0$
$f'(x) = e^x - \frac{1}{x} =$
$f'(x) > 0$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f(-x) = f(x)$$

$$\lim_{x \to \infty} f(x) = 0$$

$$f(0) = 1/2\pi$$



standard normal density)

$$f'(x) = \frac{1}{\sqrt{2\pi}} (-x) e^{-x^2/2}$$

Only critical pt $x=0$

$$f''(x) = \frac{1}{\sqrt{2\pi}} - 1 e^{-x^2/2} + \frac{1}{2\pi} (-x)(-x e^{-x^2/2})$$

$$= \frac{1}{\sqrt{2\pi}} (x^2-1) e^{-x^2/2}$$

Possible inflect^e pts: $x=-1$ or $x=+1$

$x=-1$: $f''(x) > 0$ inflect^e pt at -1

$1 > x > -1$ $f''(x) < 0$

$x=+1$ * $x > 1$, $f''(x) > 0$ inflect pt at $+1$.

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{2\sigma^2}\right)} =$$

$$g''(x) = 1/\sigma^3 \; f''\left(\frac{x+\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

1 critical pt at

$$\int \frac{dx}{1+e^{-x}} = \int \frac{e^x}{e^x(1+e^{-x})}\,dx$$

$$= \int \frac{e^x}{e^x+1}\,dx$$

$$= \int \frac{du}{u} = \ln u + c = \ln(e^x+1)$$

$a^x \quad a>0$

$a = e^{\ln a}$

If r is rational

$a^{m/n} = (e^{\ln a})^{m/n}$

$= e^{(m/n)\ln a}$

$= e^{r \ln a}$

For any real number x, $\quad a^x = e^{x \ln a}$

Domain $a^x = (-\infty, \infty)$

Range $a^x = (0, +\infty)$

$a^x = ? \; e^{x \ln a}$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$

$$= \frac{d}{dx}(x \ln a)\, e^{x \ln a}$$

$$= (\ln a) e^{x \ln a}$$

$$= \cancel{(\ln a) a^x}$$

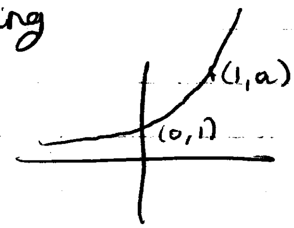
$$= (\ln a) e^{x \ln a}$$

$$= \underline{(\ln a)}\, a^x.$$

If $\ln a > 0$

- $a > 1$

$\frac{d}{dx} a^x > 0$ strictly increasing



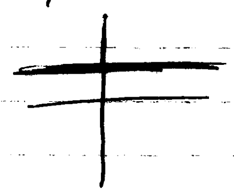
$x \to -\infty$, $x \ln a \to -\infty$

$a^x = e^{x \ln a} \to 0$

$x \to +\infty$, $x \ln a \to +\infty$

$a^x \to +\infty$.

- $a = 1$, $\ln 1 = 0$, $e^{x \ln a} = e^{x \cdot 0} = e^0 = 1$



- $a < 1$, $\ln a < 0$

$\frac{d}{dx} a^x = \underset{<0}{(\ln a)} \underset{>0}{a^x}$ strictly decreasing

$x \to -\infty$, $x \ln a \to +\infty$ $\quad a^x = e^{x \ln a} \to +\infty$

$x \to +\infty$, $x \ln a \to -\infty$ $\quad a^x = 0$

$$\left(\frac{d}{dx}\right)^2 a^x = \frac{d}{dx}\left[(\ln a)\, a^x\right]$$
$$= (\ln a)^2 a^x$$

$a \neq 1$, $(\ln a)^2 > 0$

Fact. $a^{b+c} = a^b a^c$

$a^{-b} = \frac{1}{a^b}$

$(a^b)^c = a^{bc}$

$$a^{b+c} = e^{(b+c)\ln a}$$
$$= e^{b \ln a + c \ln a}$$
$$= e^{b \ln a} e^{c \ln a}$$
$$a^{b+c} = a^b a^c.$$

$$\frac{d}{dx}\left[\frac{1}{\ln a}a^x\right] = \frac{1}{\ln a}(\ln a)\,a^x = a^x$$

10^{-3448}
10^{-2649}
10
10^{-17}

$$(a^b)^c = e^{c\ln(a^b)}$$
$$= e^{c[b\ln a]}$$
$$= e^{bc\ln a}$$
$$= a^{bc}$$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$
$$\frac{d}{dx}a^{f(x)} = \ln a\; a^{f(x)}\, f'(x)$$
$$= (\ln a)\, f'(x)\, e^{f(x)}$$

$$\int a^x dx = \frac{1}{\ln a}a^x + C$$

e.g.:
$$\int_0^2 3^x dx = \frac{1}{\ln 3}3^x\Big|_0^2$$
$$= \frac{1}{\ln 3}[9-1] = \frac{8}{\ln 3}$$

a^r r rational
a^x x real
$e^{x\ln a}$

$$f(x) = x^r = x^{m/n}$$

Domain $f = (x > 0)$ ← n even, $x > 0$

n odd,
Domain $f = [-\infty < x < +\infty]$

For any real number r
define $x^r \equiv e^{r\ln x}$, x

$$x^n = e^{n\ln x}$$
$$= (e^{\ln x})^n$$

$$x^{\sqrt{2}} = e^{\sqrt{2}\ln x}$$

$$\frac{d}{dx}x^r = \frac{d}{dx}e^{r\ln x}$$
$$= \left(\frac{r}{x}\right)e^{r\ln x}$$
$$= \left(\frac{r}{x}\right)x^r$$
$$= \cancel{\left(\frac{r x^{r-1}}{x}\right)}$$

$$\boxed{\frac{d}{dx}x^r = r x^{r-1}}$$

$a^{\log_a x} = x \quad , x > 0$

Take natural log of both sides

$\ln(a^{\log_a x}) = \ln x \;, \; x > 0$

$\log_a x (\ln a)$

$\log_a x = \frac{\ln x}{\ln a} \;, \; x > 0$

$\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$

Def: $\int x^r dx = \frac{x^{r+1}}{r+1} + C \;, \; r \neq -1$

$$\boxed{\int_0^{r \text{ real}} \frac{1}{\ln_a x} dx = \log_a x + C.}$$

$\int x \, dx = x^2/2 + C$

$\int x^3 dx = x^4/4 + C$

$\int x^{\sqrt{2}} dx = \frac{x^{\sqrt{2}+1}}{\sqrt{2}+1} + C$

$f(x)^{g(x)}$

$f(x) = (2x)^{\sin x}$

$\cancel{e^{\sin x}}$

$f(x) = e^{\sin x \ln(2x)}$

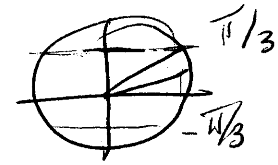
$f'(x) = \frac{d}{dx}[\sin x \ln(2x)] \, e^{\sin x \ln(2x)}$

$= [\cos x \ln(2x) + \sin x \left(\frac{2}{2x}\right)] (2x)^{\sin x}$

$f'(x) = \left[\frac{\sin x}{x} + \cos x \ln 2x\right] (2x)^{\sin x}$

a^x $a>0$, x real
x^r $x>0$, r real.

$\log_a x$, $x>0$, $a>0$, $a \neq 1$



$f(x)>0$: $f(x)^{g(x)} = e^{g(x) \ln f(x)}$

(Omit Sect. 7.4: (skip))

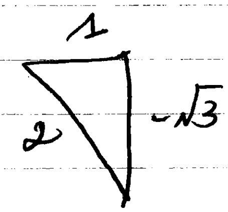
sin, cos / tan are periodic
Whenever U have periodic fct^e, U can't find its inv
because it's not one

$\sin(x+2\pi) = \sin x$ (period 2π)
$\cos(x+2\pi) = \cos x$ (period 2π)
$\tan(x+\pi) = \tan x$ (period π).

$\sin^{-1}(x) = \arcsin x$
$= \mathrm{asin}\, x$

Domain of $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\sec^{-1}$

$$\begin{cases} \sin x : [-\pi/2, \pi/2] \to [-1, 1] \\ \sin^{-1} x : [-1, 1] \to [-\pi/2, \pi/2] \end{cases}$$

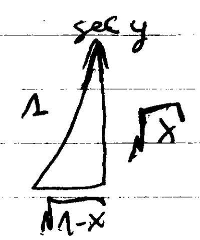


$\sin^{-1}(\sin x) = x$, $-\pi/2 \leq x \leq \frac{\pi}{2}$.

$\sin(\sin^{-1} x) = x$, $-1 \leq x \leq 1$.

e.g. $\sin^{-1}(\sin \frac{4\pi}{3}) = \sin^{-1}(\frac{-\sqrt{3}}{2}) = -\pi/3$ coz $4\pi/3$ isn't i
[-

$\sec(\sin^{-1}(\sqrt{x})) =$



$a^2 + (\sqrt{x})^2 = 1^2$
$a^2 = 1 - x$

$\sec y = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x}}$

$\sin^{-1} x = y$, $\sin y = x$ $\frac{dx}{dy} = c$

$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C.}$$

e.g.:

$$\int \frac{dx}{\sqrt{16-x^2}}\,dx = \int \frac{dx}{\sqrt{16\left(1-\left(\frac{x}{4}\right)^2\right)}}$$

$$= \frac{1}{4}\int \frac{dx}{\sqrt{1-(x/4)^2}} = \sin^{-1}\left(\frac{x}{4}\right) + C.$$

$u = \frac{x}{4}$

$du = \frac{1}{4}dx$

$$= \frac{1}{4}\int \frac{4\,du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C.$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C}$$

e.g.:

$$\int \frac{dx}{\sqrt{8x-x^2}} = ?$$

$$8x - x^2 = -(x^2-8x)$$
$$= -(x^2-8x+16-16)$$
$$= 16-(x-4)^2$$

so

$$\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}} = \frac{1}{4}\int \frac{dx}{\sqrt{1-\left(\frac{x-4}{4}\right)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u)+C$$

$u = \frac{x-4}{4}$

$du = \frac{1}{4}dx$

$$= \sin^{-1}\left(\frac{x-4}{4}\right) + C$$